S*-Iteration Approach to Strong and Δ-Convergence for Non-Expansive Type Mappings in Hyperbolic Spaces

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ABSTRACT

In this paper, we prove some strong and Δ-convergence results for non-expansive mappings through the S*-iterative process in hyperbolic spaces. Our results are an extension and generalization of existing results in the literature.

KEYWORDS

Non-expansive mappings, fixed Point, S*-iterative process, hyperbolic spaces.

1. INTRODUCTION

The concept of non-expansive mappings was introduced by [3]. It can be defined in many general settings of metric spaces. Let ß: Þ →Þ be a self-map on a nonempty subset Þ of a Banach space Ú, ß is said to be non-expansive if

$$d(ßs, ßt) \leq d(s, t) \text{ for all } s,t \in Þ \tag{1.1}$$

The notation F(ß) has been reserved for the set of fixed points of ß. The concept of Δ-convergence was introduced by Lim [14]. Previously several iterative processes have been evolved in which have been operated various numbers of steps to find the fixed points. In Banach contraction, the theorem involved a one-step iteration process known as the Picard iteration process. Some of the well-known iterative processes as Mann [15] is a one-step process, Ishikhawa [8] is a two steps process, Noor [16], S. Agrawal [2], Abbass [1] Picard -S Gursoy and Karakaya [5], Gursoy [6] and Thakur et. al. [19] are Three steps iterative process and so on. Recently, in [7] the authors introduced the following four-step iterative process, called S* iterative process in Banach Spaces. Let ß: Þ →Þ be a self-map on a nonempty subset Þ of a Banach space Ú and $\{á_n\}$, $\{é_n\}$, $\{ó_n\}$and $\{ú_n\}$are real sequences in (0,1) for all $n \geq 0$. Generate the sequence$\{h_n\}$iteratively, arbitrary $h_0 \in Þ$, by

$$\begin{cases} h_{n+1} = ß\big((1 - á_n)g_n + á_n ß g_n\big) \\ g_n = ß\big((1 - é_n)k_n + é_n ß k_n\big) \\ k_n = ß\big((1 - ó_n)j_n + ó_n ß j_n\big) \\ j_n = ß((1 - ú_n)h_n + ú_n ß h_n \qquad n \in Þ \end{cases} \tag{1.2}$$

Motivated by the above, we construct the hyperbolic space version S*-iterative Furthermore, we established Δ-convergence as well as strong convergence of the steps iterative process for non-expansive mapping in hyperbolic spaces.

2. PRELIMINARIES

In this study, we discuss the setting of hyperbolic spaces which was introduced by Kohlenbach [11], containing normed linear spaces and convex subsets and Hadamrd manifolds [17], CAT(0) spaces in the sense of Gromov[4] and Hilbert ball equipped with the hyperbolic metric [17]. In this context, we need some definitions, lemmas, and prepositions that will be used in the sequel,

Definition 1 [11] A hyperbolic space is a triple (Ú, ρ , W) where (Ú, ρ) is a metric space and $W: Ú^2 \times [0,1] \to Ú$ such that (W1) $\rho(w, W(s, t, \omega) \leq (1-\omega)\, \rho(w, s) + \omega\rho(w, t)$

(W2) $\rho(W(s, t, \omega), \rho(s, t, \sigma)) = |\omega - \sigma|\, \rho(s, t)$,

(W3) $W(s, t, \omega) = W(t, s, (1 - \omega))$,

(W4) $\rho(W(s, z, \omega), W(t, w, \omega) \leq (1 - \omega)\, \rho(s, t) + \omega\rho(z, w)$

For all s, t, w, z ϵ Ú and $\omega, \sigma \in [01]$

Definition 2 [12] A hyperbolic space (Ú, ρ, W) is called uniformly convex, if for all s, t, z ∈ Ú, r > 0 and $\varepsilon \in (0\ 2]$ there exists $\delta \in (0\ 1]$, such that

$$\left.\begin{matrix} \rho(t,s) \leq r \\ \rho(z,s) \leq r \\ \rho(t,z) \leq \varepsilon r \end{matrix}\right\} \Rightarrow \rho(W(t, z, ½), s) \leq (1 - \delta)r. \tag{2.1}$$

Definition 3 [12] A mapping μ: $(0\ \infty\ ) \times (0\ 2] \rightarrow (0\ \ 1)$ which provides $\delta = \mu(r, \varepsilon)$ for a given r > 0 and $\varepsilon \in (0\ 2]$ is well known as a modulus of uniform convexity of Ú. We call μ as a monotone if it decreases with r (for a fixed ε), i.e., for any given $\epsilon > 0$ and for any $r_2 > r_1 > 0$, we have $\mu\ (r_2, \varepsilon\ ) \leq (r_1, \varepsilon)$.

Definition 4 [12] A non-empty subset Þ of a hyperbolic space is said to be convex if W(s, t, ω) ϵ Þ for any s, t ϵ Þ and $\omega \in [0\ 1]$. If s, t ϵ Ú and $\omega \in [0, 1]$, then we use the notion $(1-\omega\ )s \oplus \omega t$ for W(s, t, ω). In [20], it is remarked that any normed space (Ú, $||.||$) is a hyperbolic space, with$(1 - \omega\ )s \oplus \omega t = (1 - \ \omega\ )s + \omega t$. Hence, the class of uniformly convex hyperbolic spaces is a natural generalization of uniformly convex Banach spaces.

Let Þ be a nonempty, closed, and convex subset of a Hyperbolic space Ú, $\{h_n\}$ a bounded sequence in Ú and s ϵ Þ, we define a function r(., $\{h_n\}$) : Ú→ $[0,\infty]$ by

r(s, $\{h_n\}$) = limsup $n\rightarrow \infty$ $\rho(s, h_n)$

An asymptotic radius of $\{h_n\}$ relative to Þ is defined by

r(Þ, $\{h_n\}$) = inf{r{s,$\{h_n\}$}) : s ϵ Þ}.

An asymptotic center of $\{h_n\}$ relative to Þ is defined by

AC(Þ, $\{h_n\}$) = {s ϵ Þ : r(s,$\{h_n\}$) = r(Þ, $\{h_n\}$) }.

The sequence$\{h_n\}$in Ú is said to Δ-convergence to s ϵ Þ if s is a unique asymptotic center of$\{k_n\}$for every subsequence$\{k_n\}$ of$\{h_n\}$. In this case, we write Δ-lim sup $n \rightarrow \infty$ $h_n = h$ and call h the Δ-lim of $\{h_n\}$.

Lemma 2.1 [13] Let Ú be a complete uniformly convex Hyperbolic space with a monotone modulus of uniform convexity μ.Then every bounded sequence $\{h_n\}$ in Ú has a unique asymptotic canter with respect to any nonempty closed convex subset Þ of Ú.

Lemma 2.2 [9] Let Ú be a complete uniformly convex Hyperbolic space with a monotone modulus of uniform convexity μ. Let s ϵ Þ and$\{á_n\}$ be a sequence in [a, b] for some a, b ϵ (0, 1). If $\{h_n\}$ and $\{g_n\}$ are sequences in Ú such that limsup $n\to\infty$ $\rho(h_n, h) \le \vartheta$, limsup $n\to\infty$ $\rho(g_n, h) \le \vartheta$ and lim $n\to\infty \rho(W(h_n, g_n, á_n), h) = \vartheta$ for some $\vartheta \ge 0$. Then, lim $n\to\infty \rho(h_n, g_n) = 0$.

3. MAIN RESULTS

Firstly, the S* -iteration process is expressed in the Hyperbolic space as follows:

Let ß: Þ →Þ be a self-map on a nonempty subset Þ of a hyperbolic space Ú and $\{á_n\}$, $\{é_n\}$, $\{ó_n\}$and $\{ú_n\}$are real sequences in (0,1) for all $n \ge 0$. Generate the sequence$\{h_n\}$iteratively, arbitrary $h_0 \in$ Þ, by

$$\begin{cases} h_{n+1} = W(ßg_n, g_n, á_n) \\ g_n = W(ßk_n, k_n, é_n) \\ k_n = W(ßj_n, j_n, ó_n) \\ j_n = W(h_n, ßh_n, ú_n) \qquad n \in Þ \end{cases} \tag{3.1}$$

Lemma 1 Let ß: Þ →Þ be a non-expansive self-mapping satisfying (1.1), where Þ is a nonempty closed & convex subset of a uniformly convex hyperbolic space Ú. Let $\{h_n\}$ be a sequence generated by (3.1); Then lim $n\to\infty$ $\rho(h, h_n)$ exists for all $h \in F(ß)$.

Proof- let $h \in F(ß)$ & $h_n \in$ Þ; since ß is non-expansive mapping, we can easily obtain that

$\rho(ßh, ß h_n) = \rho(h, h_n) \le \rho(h, h_n)$; for all $h_n \in$ Þ & $h \in F(ß)$

Thus using (2.1), we obtain that

$$\rho(j_n,h) = \rho(W(h_n, ßh_n, ú_n), h)$$

$$\le (1 - ú_n)\, \rho(h_n, h) + ú_n \rho(ßh_n, h)$$

$$= (1 - ú_n)\, \rho(h_n, h) + ú_n \rho(ßh_n, ßh)$$

$$\le (1 - ú_n)\, \rho(h_n, h) + ú_n \rho(h_n, h)$$

$$\rho(j_n, h) \le \rho(h_n, h) \tag{i}$$

using (3.1) & (i)

$$\rho(k_n,h) = \rho(W(ßj_n, j_n, ó_n), h)$$

$$\le (1 - ó_n)\, \rho(ßj_n, h) + ó_n\, \rho(j_n, h)$$

$$= (1 - ó_n)\, \rho(ßj_n, ßh) + ó_n\, \rho(j_n,h)$$

$\leq (1 - ó_n)\, \rho(j_n, h) + ó_n\, \rho(j_n, h)$

$\rho(k_n,h) \leq \rho(j_n, h)$

$\rho(k_n, h) \leq \rho(h_n, h)$ (ii)

using (3.1) & (ii)

$\rho(g_n,h) = \rho(W(ßk_n, k_n, é_n), h)$

$\leq (1 - é_n)\, \rho(ßk_n, h) + é_n\, \rho(k_n, h)$

$= (1 - é_n)\, \rho(ßk_n, ßh) + é_n \rho(k_n,h)$

$\leq (1 - é_n)\, \rho(k_n, h) + é_n \rho(k_n, h)$

$\rho(g_n, h) \leq \rho(k_n, h)$

$\rho(g_n, h) \leq \rho(h_n, h)$ (iii)

using (3.1) & (iii)

$\rho(h_{n+1},h) = \rho(W(ßg_n, g_n, á_n), h)$

$\leq (1 - á_n)\, \rho(ßg_n, h) + á_n\, \rho(g_n, h)$

$= (1 - á_n)\, \rho(ßg_n, ßh) + á_n\, \rho(g_n, h)$

$\leq (1 - á_n)\, \rho(g_n, h) + á_n\, \rho(g_n, h)$

$\rho(h_{n+1}, h) \leq \rho(g_n, h)$

$\rho(h_{n+1}, h) \leq \rho(h_n, h)$ (iv)

Thus, the sequence $\{\rho(h_n, h)\}$is bounded below & decreasing. Hence lim $n \to \infty$ $\rho(h_n, h)$ exists for all h ϵ F(ß).

Lemma 2 Let ß: Þ →Þ be a non-expansive self-mapping satisfying (1.1), where Þ is a nonempty closed & convex subset of a uniformly convex hyperbolic space Ú. Let $\{h_n\}$ be a sequence generated by (3.1). Then F(ß)≠ φ, if and only if $\{h_n\}$ is bounded & lim $n\to\infty$ $\rho(ßh_n, h_n) = 0$.

Proof- Assume that F(ß)≠ φ, & h ϵ F(ß), by lemma 1, $\{h_n\}$is bounded.

Next, we will indicate that lim $n\to \infty$ $\rho(ßh_n, h_n) = 0$

Since ß is non-expansive mapping, we have

$\rho(h, ßh_n) = \rho(ßh, ßh_n) \leq \rho(h,h_n)$ (v)

from lemma1, we achieve lim $n\to\infty$ $\rho(h_n, h)$ exists for all h ϵ F(ß)

Assume that lim $n \to \infty$ $\rho(h_n, h) = \alpha$, $\alpha > 0$. then

$\rho(k_n,h) = \rho(W(h_n, ßh_n, ó_n), h)$

$\leq (1 - v)\ \rho(h_n, h) + ó_n\ \rho(ßh_n, h)$

$= (1 - ó_n)\ \rho(h_n, h) + ó_n\ \rho(ßh_n, ßh)$

$\leq (1 - ó_n)\ \rho(h_n, h) + ó_n \rho(h_n, h)$

$\rho(k_n, h) \leq \rho(h_n, h)$

Taking limsup as $n \to \infty$

limsup $n\to\infty$ $\rho(k_n, h) \leq$ limsup $n\to\infty$ $\rho(h_n, h) = \alpha$ (vi)

From (ii) & (iv)

$\rho(h_{n+1}, h) \leq \rho(g_n, h) \leq \rho(k_n, h)$

$\rho(h_{n+1}, h) \leq \rho(k_n, h)$

Taking liminf as $n \to \infty$

$\alpha \leq$ liminf $n \to \infty$ $\rho(h_{n+1}, h) \leq$ liminf $n \to \infty$ $\rho(k_n, h)$ (vii)

From (vi) & (vii)

liminf $n \to \infty$ $\rho(k_n, h) = \alpha$, we get that

limsup $n\to\infty$ $\rho(k_n, h) \leq$ limsup $n\to\infty$ $\rho(h_n, h) = \alpha$ (viii)

It follows from lemma 2.2, (vii) & (viii)

lim $n\to\infty$ $\rho(ßh_n, h_n) = 0$

Conversely, assume that $\{h_n\}$ is bounded and lim $n\to\infty$ $\rho(ßh_n, h_n) = 0$. Let p ϵ AC(Þ, $\{h_n\}$);

Using 1.1, we have

r(ßh, $\{h_n\}$) = limsup $n \to \infty$ $\rho(ßh, h_n)$

$\leq$ limsup $n \to \infty$ $\rho(h, h_n)$; holds for all u, v ϵ Þ.

= limsup $n \to \infty$ $\rho(h, h_n)$

= r(h,{h_n}) = r(Þ, {h_n}).

That is ßh ϵ AC(Þ,{h_n}). Since Ú is uniformly convex, AC(Þ,{h_n}) is a singleton, implying that ßh = h.

Now we prove Δ-convergence theorem for non-expansive mappings in Hyperbolic space.

Theorem 3.1 Let Þ be a nonempty closed, convex subset of Ú and ß:Þ→Þ be a non-expansive mapping which satisfies condition (1.1) with F(ß)$\neq \varphi$,let{h_n}Δ-converges to a fixed point of ß

Proof- It follows from lemma 2 that {h_n} is a bonded sequence. Thus, {h_n}has a Δ-convergent subsequence. Now, we are going to show that every Δ-convergent subsequence of{h_n} has a unique Δ-limit in F(ß).

Let s and t be Δ-limits of the sequences {h_{n_j}}and { h_{n_k}} of {h_n}respectively. From lemma 2.1, we have

AC(Þ,{ h_{n_j}}) = {s} & AC(Þ,{ p_{n_k}}) = {t}

By lemma 2, we obtain that lim n→ ∞ d(h_{n_j}, ßh_n) = 0 & lim n→ ∞ d(h_{n_k}, ßh_n) = 0.

Next, we prove that s & t are fixed points of ß & s, t should be are unique, Now

ρ(ßs, p_{n_j}) ≤ ρ(ßs, ßp_{n_j}) + ρ(ßp_{n_j}, s)

≤ ρ(s, p_{n_j}) + ρ(ßp_{n_j}, s) (ix)

Implies that, r(Gu, { p_{n_j}}) = limsup $n \to \infty$ ρ(ßs, p_{n_j})

≤ limsup $n \to \infty$ ρ(ßs, ßp_{n_j}) + ρ(ßp_{n_j}, s)

` ≤ limsup $n \to \infty$ ρ(s, p_{n_j}) + ρ(ßp_{n_j}, s)

= ρ(s, p_{n_j})

= r(u,{ p_{n_j}})

The uniqueness of the asymptotic center implies ßs = s. Thus, s is a fixed point of s.

Similarly, we also have t as a fixed point of ß

Finaly, we show that s =t. Suppose s and t are distinct, by the uniqueness of an asymptotic center, we have

$\limsup n \to \infty\ \rho(h_n, s) = \limsup n \to \infty\ \rho(h_{n_j}, s)$

$< \limsup n \to \infty\ \rho(h_{n_j}, t)$

$= \limsup n \to \infty\ \rho(h_n, s)$

$= \limsup n \to \infty\ \rho(h_{n_k}, t)$

$< \limsup n \to \infty\ \rho(h_{n_k}, s)$

$= \limsup n \to \infty\ \rho(h_n, s)$

This is a contradiction. Thus s = t. Then $\{h_n\}$ Δ-converges ta a fixed point of ß.

Next, we prove some strong convergence theorems-

Theorem 3.2 Let Þ is a nonempty closed & convex subset of a uniformly convex hyperbolic space Ú & ß : Þ →Þ be a non-expansive self-mapping satisfying (1.1) with F(ß) $\neq \varphi$. Then the sequence $\{ h_n \}$ generated by the iterative scheme (3.1) converges to the point of F(ß)if and only if liminf n→ ∞d(h_n, F(ß))=0 where d(h_n, F(ß)) = inf { $\rho(h_n, h)$; h ∈ F(ß) }.

Proof- Assume that $\{h_n\}$ converges to h ∈ F(ß) so, lim n → ∞ $\rho(h_n, h) = 0$, because

$0 \leq \rho(h_n, F(ß) \leq \rho(h_n, h)$ for all h ϵ F(ß)

Therefore liminf $n \to \infty\ \rho(h_n, F(ß)) = 0$

Conversely, assume that liminf $n \to \infty\ \rho(h_n, F(ß)) = 0$ & h ϵ F(ß), from lemma1 lim $n \to \infty$ $\rho(h_n, h)$ exists for all h ϵ F(ß), therefore lim $n \to \infty\ \rho(h_n, F(ß)) = 0$ by the assumption.

Now it is enough to show that $\{h_n\}$ is Cauchy sequence in ß

Therefore lim$n \to \infty\ \rho(h_n, F(ß)) = 0$, for a given $\varepsilon > 0$ there exists $m_0 \epsilon$ N such that for all n $\geq m_0$

$\rho(h_n, F(ß)) < \varepsilon/2$

inf { $\rho(h_n, h; h\ \epsilon\ F(ß)\ \} < \varepsilon/2$

In particular, inf { $\rho(h_{m_0}$, h; h ϵ F(ß) } $< \varepsilon/2$, therefore there exists h ϵ F(ß) such that

$\rho(h_{m_0}$, h) $< \varepsilon/2$

Now for m, n $\geq m_0$

$\rho(h_{m+n}, h) \leq \rho(h_{m+n}, h) + \rho(h_n, h)$

$\leq \rho(h_{m_0}, h) + \rho(h_{m_0}, h)$

$= 2\, \rho(h_{m_0}, h)$

$\rho(h_{m+n}, h) < \varepsilon$

Thus, {h_n}is a Cauchy sequence in ß, since ß is closed there is a point q ϵ ß such that lim $n \to \infty$ h_n = q. Now lim $n \to \infty$ $\rho(h_n$, F(ß)) = 0. gives that ρ(q, F(ß)), that is q ϵ F(ß).

Theorem 3.3 Let Þ is a nonempty closed & convex subset of a uniformly convex hyperbolic space Ú with monotone modulus of uniform convexity μ & ß: Þ → Þ be a non-expansive self-mapping with F(ß)≠ φ. Suppose that either is compact or is Semi-compact. Then the sequence{p_n} generated by the iterative scheme (3.1) converges strongly to a fixed point of ß.

Proof- Note that the condition(I) in [18] is weaker than both the compactness of Þ and the semi-compactness of the non-expensive mapping; therefore we have the result by the below theorem.

Condition(I) was introduced by Senter & Dotson [18] as a requirement for mapping which is defined as the following

A mapping ß: Þ → Þ is said to satisfy condition (I). If there exists a non-decreasing function g: $R_+ \to R_+$ with g(0) = 0 & g(t) $>$ 0, for all t $>$ 0 such that ρ(u, Vu) ≥ g(ρ(u, F(V)), for all u ϵ Y. Here R_+ denotes the set of all non-negative real numbers.

Now we prove a strong convergence result using condition(I)

Theorem 3.4 Let Þ be a nonempty closed, convex subset of Ú and ß: Þ → Þ be a non-expansive self-mapping which satisfies condition (I). Then the sequence{p_n} generated by (3.1) converges strongly to a fixed point of ß

Proof- we proved the following in Lemma 2

$\lim n\rightarrow\infty\ \rho(ßh_n, h_n) = 0$ (3.2)

Using conditions (I) & (3.2), we get

$0 \leq \lim n \rightarrow \infty\ g(\rho(h_n, F(ß)) \leq \lim n \rightarrow \infty\ \rho(ßh_n, h_n) = 0$ implies $\lim n \rightarrow \infty\ g(\rho(h_n, F(ß)) = 0$.From $g: R_+ \rightarrow R_+$ with $g(0) = 0$ & $g(t) > 0$, for all $t > 0$ we have

$\lim n \rightarrow \infty\ \rho(h_n, F(ß) = 0$

By applying Theorem 3.2, we obtain the desired result; therefore, the sequence $\{h_n\}$ converges strongly to a fixed point of ß.

4. CONCLUSION

In this work, We consider a uniformly convex hyperbolic space, which is a more general framework that encompasses uniformly convex Banach spaces as a special case. Our results extend the corresponding results of S. Hassan [7] & S. H. Khan [10] for non-expansive type mappings through the S*-iterative process from Banach spaces to the general setting of hyperbolic spaces.

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