A New common fixed-point theorem for weakly compatible mappings in G-fuzzy metric space using generalized contractive conditions.

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Abstract: -In this paper , we prove a new common fixed point theorem for weakly compatible mapping satisfying the (E.A.) property in G-fuzzy metric spaces under a generalized contractive condition involving an altering distance function. The results generalize and extend several known fixed point theorems in fuzzy metric spaces.An example is provided to illustrate the validity of the main theorem.

Keywords: - G-Fuzzy Metric Spaces, weakly compatible Mappings, (E.A.) Property, generalized contractive condition, common fixed point **.**

1. Introduction–

Fixed point theory in fuzzy metric space has been a topic of significant interest since the foundational work of George and Veeramani (1994)[4], who introduced a framework based on continuous t-norms. Later, Gregori and Romaguera (2003 [5] and others further enriched this structure by introducing generalized fuzzy metric Spaces such as G- fuzzy metric spaces, which provide a broader analytical framework to study convergence and fixed Points.

In recent years, several researchers have developed fixed point theorems in various generalized fuzzy settings. For instance, Sharma et al. (2021) [9] and Abbas & Rhoades (2023)[2] Studied Common fixed point results in intuitionistic and K-fuzzy metric spaces without requiring Continuity . Srivastava and minana (2024) [10] introduced vector- valued fuzzy metric Spaces while Miñana et al. (2024 [7] proposed notions of P- completeness in fuzzy metric spaces to handle generalized Contractive mappings. These results not only extend classical theorems but also address practical problems involving uncertainties and non linear mappings.

Weak Compatibility and the (E.A.) property, introduced by Jungck (1986)[6] and further studied in fuzzy settings by Tahiri and Nuino (2024)[[11] Play a crucial role in establishing the existence of common fixed points.Many recent contributions, such as Yue et al. (2023)[12] have generalized Classical Contraction Conditions via implicit relations or altering distance functions.

Motivated by these developments, we propose a new Common fixed point theorem for weakly compatible mappings in G- fuzzy metric spaces using generalized Contractive Conditions defined via an implicit function pair (ψ, φ) . Our result generalizes several known results and provides a unified frame work that encompasses a wider class of mappings. An example is included to validate the applicability of our main theorem.

2. Preliminaries

In this section we recall some basic definitions and notations which will be used throughout the paper.

2.1 Continuous t-norm

A binary operation * on the interval [0,1] is called a Continuous t-norm

if it satisfies the following

- a*1 = a for all a in [0,1],
- a* b = b*a (Commutative).
- a* (b*c) = (a*b) *c (Associative),
- * is Continuous and non-decreasing,
- a * b ≤ min {a,b}

Example: The minimum operation min {a,b} and product a.b are both continuous t-norms .

2.2 G-Fuzzy Metric Space [4]

Let x be a non-empty set, and let T:[0,1] x [0,1] →[0,1] be a continuous t-norm. A mapping G:X x X x (0, ∞) → (0,1)

1. G (x,y,t) > 0,
2. G (x,y,t)=1 if and only if x=y,
3. G (x,y,t) =G (y,x,t),
4. G (x,z,t + s) ≥ T(G (x,y,t) , G(y,z,s)),
5. For all x,y ∈ x, the function
   t →G(x,y,t) is continuous and non decreasing on (0,∞).
   Then the triple (X,G,T) is Called a G- fuzzy Metric Space.

Remark: The notation (X,G,T) is Used to represent a G-fuzzy metric space in order to emphasize all the essential components that define its structure. Here,

- X is a non empty set of elements under consideration,
- G: X x X x (0,∞) → [0,1] is a G-Fuzzy Metric that Measures the degree of closeness between points in X, and
- T: [0,1] X[0,1] → [0,1] is a continuous t-norm that governs the generalized triangular inequality.

This notation is analogous to the classical fuzzy metric space (X,M,*) where the t-norm is also an essential part of the structure. Including T in the notation ensures that the specific type of triangle inequality being used is explicitly identified.

2.3 Weakly Compatible Mappings [3]

Let f and g be two self-Mappings on a set X .Then f and g are said to be weakly compatible if they commute at their coincidence point, i.e

f(x)=g(x)
then f(g(x)) = g(f(x))
for all x ∈ x such that f(x) =g(x).

2.4 (E.A.) Property [1]

A pair of self mappings (f,g) on a set X is said to satisfy the (E.A.) Property if there exists a sequence $x_n \subset X$ such that

$\lim_{n\to\infty} f(x)_n = \lim_{n\to\infty} g(x_n) = z$ for some z∈x

2.5 Coincidence point:

A point x ∈ X is said to be a Point of coincidence of the mappings f and g if
f(x)=g(x)

2.6 Common fixed point.

A Point x ∈ X is called a Common fixed point of f and g if
f(x)=g(x)=x

Note "In order to establish our main fixed point results, we first introduce the notion of implicit generalized Contractive Conditions in G- fuzzy metric spaces."

2.7 Implicit Generalized Contractive Condition [8]

Let (X,G,T) be a G-fuzzy metric space, and let f , g : X → X be two self- mappings.
Suppose there exist continuous functions

- Φ, Ψ:[0,1]→[0,1],and
- F: $[0,1]^5$→[0,1],

where Φ *and* Ψ are non-decreasing and satisfy the conditions

1. $\Phi(r) < r,\ \Psi(r) \le r, for\ all\ r \in (0,1)$
2. F(U,U,U,U,U)≥ $\Psi(U), for\ all\ U \in (0,1)$

   If for every x ,y ,∈,X and t > 0
   The following inequality holds:

$$\Phi(G(fx, gy, t)) \le F(\Psi\ (G(x, y, t), \psi(G(x, fx, t), \Psi\big(G(y, gy, t)\big), \Psi\big(G(x, gy, t)\big), \Psi(G(y, fx, t))$$

Lemma 1: Convergence property in G-Fuzzy Metric Space

Let (X,G,T) be a G-Fuzzy Metric Space. If {x_n} is a sequence in X such that $\lim_{n\to\infty} G(x_n, x_{n+1}, t)$ and $\lim_{n\to\infty} G(x_n$,x,t) = for all t>0

Lemma 2: - coincidence point under (E.A.) Property and continuity

Let f and g be self-mappings on a G-Fuzzy metric space (X , G, T).Suppose the pair (f,g) satisfies the (E.A) Property and g is continuous. Then f ang g have a coincidence point i.e. , there exist.

$$Z\in X\ such\ that\ f(z) = g(z)$$

Example: A G - Fuzzy Metric space with weakly compatible Mappings Satisfying the E.A. Property.

Let X = R, and define

G:X x X x (0,∞)→ [0,1] by

G (x, y, t) = $\frac{t}{t+|x-y|}$.

Define mappings f , g : x → x by

f(x) = $\frac{x}{2}$, g(x) = x.

- The pair (f,g) Satisfies the (E.A.) property, Since f (x_n) = $\frac{x_n}{2} \to 0$ and g(x_n) = $x_n \to o$ as $x_n \to o$
- The pair is weakly compatible, since if f(z)=g(z) , then z = $\frac{z}{2}$ $\Rightarrow z = o$ and at that Point f(g(0)) = f(0) = 0 = g(f(0))

Hence the mappings satisfy the Conditions of the main theorem.

3. Main Theorem

Let (X, G,T) be a G-fuzzy metric space. Let f, g : x→x be two self-mappings satisfying.

1. The pair (f, g) satisfies the (E.A.) property; that is, there exist a sequence x_n in X Such that
   $\lim_{n\to\infty} G(x_n, fx_n, t)) = \lim_{n\to\infty} G(x_n, gx_n, t)) = 1$ for all t>o.
2. g(x) is closed in X.
3. f and g are weakly compatible; that is ,if fx = gx then f(g(x)) = g(f(x))
4. There exist functions Φ, Ψ: [0,1]→[0,1] and a continuous function

$$F: [0,1]^5 \to [0,1]$$

such that.

a) Φ is continuous and satisfies Φ(r) < r fox all r∈(0,1),

b) ψ is Continuous and satisfies. ψ (r) ≤ r for all r ∈ (0,1),

c) F is continuous and satisfies F(U,U,U,U,U) ≥ ψ (U) for all U ∈ (0, 1)

d) for all x,y ∈ X and $t > 0$ following inequality holds:

Φ (G(fx,gy,t)) ≤ F (ψ(G(x,y,t)), (ψ (G(x,fx,t)),

Φ (G(y,gy,t)) , ψ (G(x.gy,t)), ψ (G (y, fx,t))

Then f and g have a unique common fixed point in X

Proof:- Let (X, G. T) be a G-fuzzy metric space and let f, g: x → x satisfy the Conditions of the theorem.

by (E.A.) property, there exists a sequence{x_n} in X such that:

$\lim_{n\to\infty} G(fx_n, gx_n, t)$ = 1 for all $t > 0$ ………………(1)

since g(x) is closed, the sequence {g(x_n)} ⊆ g(x) has a limi point U∈X such that:

$$\lim_{n\to\infty} gx_n = U, \text{ and hence } U \in g(x).$$

So there exists z∈X Such that g (z) = U .

Also, from (1), $\lim_{n\to\infty} fx_n$=U

We now show that U is a Coincidence point of f and g, i.e, f(z) = g(z) = U.

from continuity of G and the Condition (d), fox all n∈N and t<o, we have:

Φ (G(fx_n,gx_n,t)) ≤F (ψ(G (x_n,x_n,t), ψ(G(x_n,fx_n,t)) , ψ(G(x_n,gx_n,t)) , ψ(G(x_n,gx_n, t)) ,ψ(G(x_n,fx_n,t))

using the properties of a G-fuzzy metric space (Symmetry and identity)

G (x_n, x_n ,t) = 1 so ψ(G(x_n, x_n ,t) =$\psi(1)$.

Since $fx_n \to U$ and $gx_n \to U$, then using the Continuity of G we have :

G (X_n, fx_n ,t) → G (U,U,t) = 1

And Similarly G (x_n,gx_n,t) → 1

So as n → ∞ all arguments of F Converge to ψ(1), hence

RHS→ F(ψ(1), ψ(1), ψ(1), ψ(1), ψ(1))

but LHS→ Φ(G(fx_n,gx_n,t)) → Φ(G(U,U,t))= Φ(1)

{ since fx_n,$gx_n \to U$}

So in the limit

Φ (1) ≤ F(ψ(1), ψ(1), ψ(1), ψ(1), ψ(1))

using assumption (C):

F (U,U,U,U,U)≥ ψ (U) for all U∈ $(0,1)$

Now as G (U,U,t) = 1 and since

Φ(r)≤r and ψ(r)≤r for r ∈(0,1), we conclude: Φ (1) ≤ $\psi(1) \leq 1$

So the inequality holds in the limit as well. Now, fx_n→U and gx_n→U ⇒ F(z) = g(z) = U

(by closedness and continuity).

Hence U is a coincidence point.

Now f and g are weakly compatible

⇒f g (z) = g f (z) ⇒ f(u) = g(u)

But f(z) =U =g(z), so, f(u)= g(u) ⇒ f (f(z)=g(f(z))

⇒ f (z) is a fixed point of both f and g ⇒ common fixed point .

Uniqueness :-Suppose there exist another Common fixed point V≠U Such that f(v) = g(v) = V Then applying the contractive Condition on x=z , y = v , and using f(z) = g(z) = U and f(v) = g(v) = v, we get.

Φ (G (u,v,t)) ≤ F (ψ (G(z,v,t)) , ψ (G(z,u,t)) , ψ (G(v,v,t)) , ψ (G(z,v,t)) , ψ (G(v,u,t)).

Since G is fuzzy metric and Symmetric, and using Contractive Condition assumptions, we deduce:

Φ (G(u,v,t)) ≤ ψ (G(u,v,t))< (G(u,v,t))

⇒ (G(u,v,t)) < (G(u,v,t)) , Which is contradiction

Hence, U=V ⇒ Uniqueness

Hence, f and g have a unique common fixed point.

3.1 Example: Let X = R (the set of real numbers) and define the G-fuzzy metric G: RxR x(0 ,∞)→(0,1) , by:

G (x,y,t) = $\frac{t}{t+|x-y|}$, x,y ∈ R , t>0

Then (X.G,T) is a G-fuzzy metric space, where the t-norm T (a,b) = a b

Define mappings f,g :R → R by

f(x) = x/2 ,g(x)=x

Also define the functions Φ, ψ : [0,1] → [0,1] by Φ (r) = r/2, ψ= r/2

Define F: $[0,1]^5$ → [0,1] by :

F (U_1, U_2, U_3,U_4,U_5)= max {(U_1, U_2, U_3,U_4,U_5)}

we now verify that all the hypotheses of the main theorem are satisfied:

Step 1 : f(x)=g(x)

We observe that f(x) = x/2 ∈R=g(R)

Hence, f(x) ⊂ g(x).

Step 2: f and g satisfy (E.A.) property:

Let $X_n = \frac{1}{n}$ ∈x then:

- $f(X_n) = \frac{1}{2n} \to 0$,
- $g(X_n) = X_n = \frac{1}{n} \to 0$ so
  $\lim f(X_n) = \lim g(X_n) = z = 0$

  thus f and g satisfy the (E.A.) Property .

Step 3: f and g have a coincidence Point : f(x) = g(x) ⇒ x/2 = x ⇒ x =0 . So 0 is a coincidence point.

Step 4: g(x) is closed in X since g(x)= x , we have g(x)=R,Which is closed in R.

Step 5: f and g are weakly compatible : At Coincidence point x=0

f(g(0)) = f(0) = 0, g f(0) = g(0) = 0 ⇒ f(g (0) = g(f(0)).

Step 7: verify contractive condition . we check the inequality ,

Φ (G(fx,gy,t)) ≤ F($\psi(G(x,y,t), \psi(G(x,fx,t), \psi(G(y,gy,t), \psi(G(x,gy,t),$ $\psi(G(y,fx,t)),$

Let us Compute each term with Values:

f(x)=x/2,g(y)=y , fx=x/2, gy=y

Now Calculate

L.H.S:- Φ (G(fx gy.t)) = $\frac{1}{2}.\frac{t}{t+|\frac{x}{2}-y|}$

R.H.S:- 1. $\psi((G(x,y,t))= \frac{1}{2}.\frac{t}{t+|x-y|}$

2. $\psi((G(x,fx,t))= \frac{1}{2}.\frac{t}{t+|x-\frac{x}{2}|} = \frac{1}{2}.\frac{t}{t+|\frac{x}{2}|}$

3. $\psi((G(y,gy,t) = \frac{1}{2}.\frac{t}{t+|y-y|} = \frac{1}{2}.1 = \frac{1}{2}$

4. $\psi(\big(G(x,gy,t)\big)= \frac{1}{2}.\frac{t}{t+|x-y|}$

5. $\psi((G(y,fx,t) = \frac{1}{2}.\frac{t}{t+|y-\frac{x}{2}|}$

Hence ,

$\frac{1}{2}.\frac{t}{t+|\frac{x}{2}-y|} \leq \max\left\{\frac{1}{2}.\frac{t}{t+|x-y|}, \frac{1}{2}.\frac{t}{t+|x|/2}, \frac{1}{2}, \frac{1}{2}.\frac{t}{t+|x-y|}, \frac{1}{2}.\frac{t}{t+y-x/2}\right\}$ ∀ x,y ∈R , t<0

∴ The contraction condition holds

All conditions of the main therem are satisfied, hence f and g have a unique common fixed point in the G-fuzzy metric space.

3.2 Corollary

Let (X,G,T) be a G-fuzzy metric space . Let f:x→x be a self mapping satisfying the following conditions:

There exist functions Φ, ψ : [0.1] →[0,1] such that

- Φ is continuous and strictly increasing, with Φ(r) < r for all r ∈ (0,1),

- Ψ is continuous and $\psi(r) \leq r$ for all $r \in (0,1)$, and for all x,y,∈X and $t > 0$, the following implicit contractive condition holds
  Φ (G(fx,fy,t))≤max{ $\psi\big(G(x,y,t)\big)$, $\psi\big(G(x,fx,t)\big)$,
  $\psi\big(G(y,gy,t)\big), \psi\big(G(x,fy,t)\big), \psi(G(y,fx,t))\}$

suppose that f satisfies the (E.A.) property and the set f(X) is closed Then f has a Unique fixed point in X.

4.Conclusion: - In this paper, we have established a new common fixed point theorem for weekly compatible mappings in G- fuzzy metric spaces under an implicit generalized Contractive Condition. By utilizing the (E.A.) property along with the Closedness of image sets, we have extended and unified several Classical fixed point results in fuzzy metric theory.

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