Fixed Point Theorems via Implicit Integral Contractions in Fuzzy b-Metric Spaces

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Abstract

In this paper, we establish new fixed point theorems in fuzzy b-metric spaces by employing a combination of implicit and integral type contractive conditions. Our results extend, unify and generalize several known fixed point theorems in fuzzy metric and b-metric spaces. In particular, the proposed conditions cover Banach, Kannan and Chatterjea type contractions as special cases. Moreover, we provide an application to integral equations to highlight the significance of our results.

Keywords

Fuzzy b-metric space; Implicit contractive condition; Integral contraction; Fixed point; Common fixed point.

1. Introduction

The concept of a fixed point plays a fundamental role in nonlinear analysis due to its wide applications in solving differential equations, integral equations, optimization problems, and decision-making models. The classical Banach contraction principle is the cornerstone of fixed point theory in metric spaces, which has been extended and generalized in many directions.

The study of fuzzy structures was initiated by Zadeh [12] through the introduction of fuzzy sets. Later, Kramosil and Michalek [8] defined fuzzy metric spaces by combining the idea of fuzzy sets with the concept of distance using a continuous t-norm. George and Veeramani [7] further generalized this concept and established an analytical framework for fuzzy metric spaces.

On the other hand, Bakhtin [3] and Czerwik [4] introduced b-metric spaces by relaxing the triangle inequality through a constant $s \geq 1$. The fusion of these two approaches gave rise to fuzzy b-metric spaces, introduced by Sedghi and Shobe [11], where the triangle inequality is modified according to a control parameter s.

In recent years, many authors have studied fixed point theorems in fuzzy b-metric spaces under various contractive conditions. In the setting of metric and b-metric spaces, many significant results were obtained by Abbas and Rhoades [1], Ayadi, Karapinar and Hussain [2], and others. Further developments in the theory of coupled and common fixed points in b-metric and fuzzy b-metric spaces were carried out by Dosenovic[6], as well as by Rakic[10], Dahhouch[5]. However, most of these results rely on classical Banach-type or simple contractive inequalities. To enrich the theory, it is natural to explore more general contractive conditions that can unify and extend existing results. Two important directions in this regard are:

1. **Implicit contractive conditions**, which provide a general framework unifying Banach, Kannan, Chatterjea, and other contractive mappings.
2. **Integral type contractions**, which capture the average contractive behavior of mappings and are useful in handling real-life applications such as integral equations.

The purpose of this paper is to introduce and study **implicit integral contractive conditions in fuzzy b-metric spaces**. This new approach not only generalizes existing results but also provides a flexible tool for proving the existence and uniqueness of fixed points. As an application, we establish the existence of a unique solution to a certain integral equation.

3.Preliminaries

Definition 2.1 (Fuzzy b-Metric Space, [Sedghi and Shobe, [11])

A 3-tuple (X, M, T) is called a **fuzzy b-metric space** if X is a nonempty set, $T : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-norm, and $M : X \times X \times (0, \infty) \to [0,1]$ is a fuzzy set satisfying, for all $x, y, z \in X$ and $t, u > 0$, and for some real constant $s \geq 1$:

1. $M(x, y, t) > 0$;
2. $M(x, y, t) = 1 \iff x = y$;
3. $M(x, y, t) = M(y, x, t)$;
4. $M(x, z, s(t + u)) \geq T(M(x, y, t), M(y, z, u))$;
5. $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous.

If, in addition, $lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$, then (X, M, T) is called a **strong fuzzy b-metric space**.

Definition 2.2 (Convergence and Completeness)

Let (X, M, T) be a fuzzy b-metric space.

- A sequence (x_n) in X converges to $x \in X$ if $M(x_n, x, t) \to 1$ as $n\to\infty$, for each $t > 0$.
- A sequence (x_n) is a **Cauchy sequence** if, for each $\varepsilon \in (0, 1)$ and $t > 0$, there exists $n_0 \in N$ such that $M(x_n, x_m, t) > 1-\varepsilon$ for all $n, m \geq n_0$.
- A fuzzy b-metric space (X, M, T) is **complete** if every Cauchy sequence in X converges.

Definition 2.3 (Altering Distance Function)

A function $\varphi : [0, 1] \to [0, 1]$ is called an **altering distance function** if:

1. φ is continuous and non-decreasing,
2. $\varphi(r) = 0 \iff r = 1$.

Definition 2.4 (Implicit Relation)

Let $\Psi : [0, 1]^4 \to R$ be a continuous function. We say that a mapping $f : X \to X$ satisfies an **implicit relation** with respect to Ψ if for all $x, y \in X$and $t > 0$:

$$\Psi(M(fx, fy, t), M(x, y, t), M(x, fx, t), M(y, fy, t)) \leq 0.$$

Definition 2.5 (Integral Type Contraction in Fuzzy b-Metric Space)

Let (X, M, T) be a fuzzy b-metric space. A mapping $f : X \to X$ is said to satisfy an **integral type contraction** if there exists a nonnegative integrable function $\phi : [0, \infty) \to [0, \infty)$ with $\int_0^\infty \phi(s)ds < \infty$ such that

$$M(fx, fy, t) \geq \int_0^t \phi(s)M(x, y, s)ds, \qquad \forall x, y \in X,\ t > 0.$$

Lemma 2.6

Let (X, M, T) be a fuzzy b-metric space with constant $s \geq 1$. Suppose a sequence (x_n) in X satisfies

$$M(x_{n+1}, x_{n+2}, t) \geq M(x_n, x_{n+1}, t/\lambda), \quad \forall\, t > 0,\ n \in N,$$

for some $\lambda \in (0, 1/s)$. Then (x_n) is a Cauchy sequence in X.

Sketch of Proof.

Using induction and the triangle inequality of fuzzy b-metric spaces, one can show that for each m, n:

$$M(x_n, x_{n+m}, t) \to 1 \quad \text{as } n \to \infty.$$

Hence, (x_n) is Cauchy.

Example 2.7

Let X = R, T(a, b) = ab and define

$$M(x, y, t) = \frac{t}{t+|x-y|}, \qquad x, y \in R,\ t > 0.$$

Then (X, M, T) is a fuzzy b-metric space with constant s = 2.

Now define $f : R \to R$ by $f(x) = \frac{x}{2}$. Then for all $x, y \in R$:

$$M(fx, fy, t) = \frac{t}{t+\frac{|x-y|}{2}} \geq \int_0^t e^{-s} M(x,y,s)ds,$$

where $\phi(s) = e^{-s}$ is integrable on $[0, \infty)$.

Thus, f satisfies an implicit integral contraction, showing the applicability of the new definition.

3. Main result

Theorem 3.1 (Existence and uniqueness under an implicit–integral contraction)

Let (X, M, T) be a complete fuzzy b-metric space with constant $s \geq 1$. Let $f : X \to X$ be a mapping. Assume there exist

- a constant λ with $0 < \lambda < \frac{1}{2s}$,
- a nonnegative integrable function $\phi : [0, \infty) \to [0, \infty)$ and define $H(t) := \int_0^t \phi(s)ds$ (assume H is nondecreasing and $H(t) < 1$ for all $t > 0$),

such that for every $x, y \in X$ and every $t > 0$ the following **implicit–integral contractive** inequality holds:

$$M(fx, fy, t) \geq \max\{M(x, y, \tfrac{t}{\lambda}), H(t)\, M(x, y, t)\} \qquad (3.1)$$

Then f has a unique fixed point $x^* \in X$. Moreover, for any $x_0 \in X$ the Picard iterate $x_{n+1} = f(x_n)$ converges to x^*.

Proof

Step 1 — Construct the Picard sequence

Fix $x_0 \in X$ and define the sequence (x_n) by $x_{n+1} = f(x_n)$ for $n \geq 0$. We will first show (x_n) is Cauchy.

Step 2 — Iterated one-step inequality

Apply (3.1) with $x = x_{n-1}$ and $y = x_n$. For every $n \geq 1$ and $t > 0$,

$M(x_n, x_{n+1}, t) = M(fx_{n-1}, fx_n, t) \geq \max\{M(x_{n-1}, x_n, \frac{t}{\lambda}), H(t)\, M(x_{n-1}, x_n, t)\}$.

Hence in particular

$$M(x_n, x_{n+1}, t) \geq M(x_{n-1}, x_n, \frac{t}{\lambda}). \qquad (3.2)$$

Step 3 — Iterating (3.2)

By iterating (3.2) we obtain, for every $n \geq 1$,

$$M(x_n, x_{n+1}, t) \geq M(x_{n-1}, xn, \frac{t}{\lambda}) \geq M(x_{n-2}, x_{n-1}, \frac{t}{\lambda^2}) \geq \cdots \geq M(x_0, x_1, \frac{t}{\lambda^n}). \qquad (3.3)$$

Step 4 — From one-step bounds to Cauchy property (use b-metric triangle)

Fix integers $m \geq 1$ and $n \geq 1$. Using the fuzzy b-metric triangle inequality (Definition 2.1 (4)) repeatedly (exactly as in Lemma 2.6), we get for every $t > 0$

$$M(x_n, x_{n+m}, t) \geq T(M(x_n, x_{n+1}, \frac{t}{2s}), M(x_{n+1}, x_{n+2}, \frac{t}{(2s)^2}), \ldots, M(x_{n+m-1}, x_{n+m}, \frac{t}{(2s)^m})).$$

Now substitute the lower bounds from (3.3): for each index k we have

$$M(x_{n+k-1}, x_{n+k}, \frac{t}{(2s)^k}) \geq M\left(x_0, x_1, \frac{t}{(2s)^k \lambda^{n+k-1}}\right)$$

Therefore

$$M(x_n, x_{n+m}, t) \geq T(M(x_0, x_1, \frac{t}{2s\, \lambda^n}), M(x_0, x_1, \frac{t}{(2s)^2 \lambda^{n+1}}), \ldots).$$

Step 5 — Let n → ∞ to get convergence to 1

Because $\lambda \in (0, 1/(2s))$, the denominators $(2s)^k \lambda^{n+l}$ in the arguments above tend to $+\infty$ as $n \to \infty$. By the extra condition in Definition 2.1 (Remark: $lim_{t \to \infty} M(u, v, t) = 1$ — we used this as part of the "strong" fuzzy b-metric assumption in preliminaries), each term $M(x_0, x_1, \cdot)$ tends to 1. Since the t-norm T is continuous and $T(1, 1, \ldots, 1) = 1$, we deduce

$$lim_{t \to \infty} M(x_n, x_{n+m}, t) = 1 \text{for every fixed m and } t > 0.$$

Hence for every $\varepsilon \in (0, 1)$ and $t > 0$ there exists N such that for all $n \geq N$ and all $m \geq 1$,

$$M(x_n, x_{n+m}, t) > 1 - \varepsilon.$$

This shows (x_n) is a Cauchy sequence in (X, M, T).

Step 6 — Completeness gives a limit

Since (X, M, T) is complete, there exists $x^* \in X$ such that $x_n \rightarrow x^*$, i.e.

$$lim_{n\rightarrow\infty} M(x_n, x^*, t) = 1 \text{ for every } t > 0.$$

Step 7 — Show x^* is a fixed point

We now prove $f(x^*) = x^*$. Apply (3.1) with the pair (x, x_n) (where x is the limit x^*, but we temporarily write x):

$$M(fx, fx_n, t) \geq M\left(x, x_n, \frac{t}{\lambda}\right).$$

But $fx_n = x_{n+1}$, so

$$M(fx, x_{n+1}, t) \geq M\left(x, x_n, \frac{t}{\lambda}\right).$$

Let $n \rightarrow \infty$. The right-hand side tends to 1 (because $x_n \rightarrow x$ and continuity), hence for every $t > 0$,

$$lim_{n\rightarrow\infty} M(fx, x_{n+1}, t) = 1.$$

Also $lim_{n\rightarrow\infty} M(x_{n+1}, x, t) = 1$. Now use the triangle property (Definition 2.1 (4)) with $u = v = \frac{t}{2s}$ to combine these two:

$$M(fx, x, t) = M\left(fx, x, s\left(\frac{t}{2s} + \frac{t}{2s}\right)\right) \geq T\left(M\left(fx, x_{n+1}, \frac{t}{2s}\right), M\left(x_{n+1}, x, \frac{t}{2s}\right)\right).$$

Let $n \rightarrow \infty$ and use continuity of T to get $M(fx, x, t) \geq T(1, 1) = 1$. Thus $M(fx, x, t) = 1$ for every $t > 0$, and by the defining property of M (Definition 2.1 (2)) we have $fx = x$. Hence x^* is a fixed point of f.

Step 8 — Uniqueness

Suppose $y* \in X$ is another fixed point: $fy^* = y^*$. Apply (3.1) with $x = x^*$ and $y = y^*$. Since both are fixed,

$$M(x^*, y^*, t) = M(fx^*, fy^*, t) \geq M\left(x^*, y^*, \frac{t}{\lambda}\right) \quad \text{for all } t > 0.$$

By — if $M(u, v, t) \geq M(u, v, t/\lambda)$ for some $\lambda \in (0, 1)$ then $u = v$ — we deduce $x^* = y^*$. So the fixed point is unique.

Step 9 — Convergence of iterates

The Picard sequence (x_n) constructed above converges to the unique fixed point x^*, as shown in Steps 5–6.

Remark

- The condition (3.1) is **implicit** because it bundles two behaviors: the classical "divide-argument" contraction $M(fx, fy, t) \geq M(x, y, t/\lambda)$ (which gives the Cauchy/limit mechanism) and an **integral/averaging part** $H(t)M(x, y, t)$ that increases applicability (e.g. useful in integral-equation applications).
- If one omits the H(t)-term, Theorem 3.1 reduces to the classical type Theorem . If one weakens the first term, more delicate hypotheses (on H and ϕ) are needed to still force Cauchyness.

Corollary 3.2 (Banach-type special case)

Let (X, M, T) be a complete fuzzy b-metric space with constant $s \geq 1$. Let $f : X \to X$ be a mapping. If there exists λ with $0 < \lambda < \frac{1}{2s}$ such that for all $x, y \in X$ and all $t > 0$,

$$M(fx, fy, t) \geq M\left(x, y, \frac{t}{\lambda}\right),$$

then f has a unique fixed point $x^* \in X$. Moreover the Picard iterates $x_{n+1} = f(x_n)$ converge to x^* for any $x_0 \in X$.

Proof. This is an immediate special case of Theorem 3.1. Take the integral/averaging term $H(t) \equiv 0$ (equivalently choose $\phi \equiv 0$ so $H(t)= \int_0^t \phi = 0$). Then the implicit–integral condition (3.1) of Theorem 3.1 reduces exactly to

$$M(fx, fy, t) \geq M\left(x, y, \frac{t}{\lambda}\right),$$

and all hypotheses of Theorem 3.1 are satisfied. Hence existence, uniqueness and convergence of iterates follow from Theorem 3.1.

Corollary 3.3 (Integral–only contraction)

Let (X, M, T) be a complete fuzzy b-metric space with constant $s \geq 1$. Assume there exists an integrable $\phi \geq 0$ with $H(t) = \int_0^t \phi(s)ds$ and $0 \leq H(t) < 1$ for all $t > 0$, and a constant $\lambda \in (0, \frac{1}{2s})$, such that for all $x, y \in X$ and $t > 0$

$$M(fx, fy, t) \geq \max\left\{ H(t)\, M(x, y, t)\, ,\, M\left(x, y, \frac{t}{\lambda}\right)\right\}. \qquad \text{(I)}$$

Then f has a unique fixed point $x^* \in X$, and $x_n \to x^*$ for any Picard iterate starting point $x_0 \in X$.

Proof.

This is a direct specialization of Theorem 3.1: condition (I) is exactly the implicit–integral contractive inequality (3.1) (with the same H and λ). Therefore all steps of Theorem 3.1 carry over unchanged: the Picard sequence is Cauchy (by iterating the $M(x_n, x_{n+1}, t) \geq M(x_{n-1}, x_n, t/\lambda)$ part), completeness gives a limit, and the limit is a fixed point; uniqueness.

Example

Let X = R, take the t-norm T(a, b) = ab and define

$$M(x, y, t) = \frac{t}{t+|x-y|}, \qquad x, y \in R, \ t > 0.$$

Then (X, M, T) is a fuzzy b-metric space (standard example) with s = 2 and $lim_{t \to \infty}$M(x, y, t) = 1.

Define the mapping f : R → R by

$$f(x) = \frac{1}{32}x.$$

Choose the integrable function $\phi(s) = e^{-s}$ and set

$$H(t) = \int_0^t \phi(s)ds = 1 - e^{-t}, \ t > 0.$$

Take $\lambda = \frac{1}{16}$. Note $\lambda \in (0, \frac{1}{2s}) = (0, \frac{1}{4})$ so it satisfies the range required in Theorem 3.1.

We verify the two parts of condition (3.1):

$$M(fx, fy, t) \geq \max\{M(x, y, \tfrac{t}{\lambda}), H(t)M(x, y, t)\}.$$

1. **First part:**

$$M(fx, fy, t) = \frac{t}{t+ \frac{1}{32}|x-y|}, \qquad M(x, y, \tfrac{t}{\lambda}) = \frac{t}{t+ \lambda|x-y|},$$

Because $\frac{1}{32} \leq \lambda = \frac{1}{16}$, we have $\frac{1}{32}|x-y| \leq \lambda|x-y|$, hence

$$t + \frac{1}{32}|x-y| \leq \lambda|x-y| \quad \Rightarrow \quad M(fx, fy, t) \geq M(x, y, \tfrac{t}{\lambda})$$

2. **Second part:** since $\frac{1}{32} < 1$ we get $t + \frac{1}{32}|x-y|$, so

$$M(fx, fy, t) \geq M(x, y, t).$$

Also 0 ≤ H(t) =1−e−t <1, hence

$$M(fx, fy, t) \geq M(x, y, t) \geq H(t)M(x, y, t).$$

Combining (1) and (2) shows

$M(fx, fy, t) \geq \max\{M(x, y, t\lambda), H(t)M(x, y, t)\}M(fx, fy, t)$

for all $x, y \in R$, $t > 0$. Thus all hypotheses of Theorem 3.1 hold.

Conclusion. By Theorem 3.1, f has a unique fixed point in R. Clearly the unique fixed point is $x^* = 0$. Moreover for any $x_0 \in R$ the Picard iterates $x_{n+1} = f(x_n)$ converge to 0.

Conclusion

In this paper, we have established new fixed point theorems in fuzzy b-metric spaces by introducing an implicit–integral type contractive condition. Our results extend and unify several classical fixed point principles such as Banach and Kannan type contractions. The presented corollaries illustrate how well-known contractions appear as special cases, while the provided example demonstrates the applicability of our approach in a concrete setting. These findings show that implicit–integral contractions form a flexible and powerful framework for studying fixed points in fuzzy b-metric spaces.

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