MELKERSSON CONDITION AND SERRE SUBCATEGORIES

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Abstract

Let R be a noetherain ring, a be an ideal of R,M and N R-modules. Let δ be a Serre subcategory of the category of R-module . We define δ- $dim(M)$ and δ-$dim(M,N)$ .We fine some conditions under which $H_a^i(M)$ and $H_a^i(M,N)$ belong δ for all $i \geq 0$.

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1. INTRODUCTION

Throughout R is a commutative noetherian ring. As a general reference to homological and commutative algebra we use [1]. Recall that a class of R-modules is a serre subcategory of the category of R-modules when it is closed under taking submodules, quotients and extensions. Always, δ stands for a serre subcategory of the category of R-modules. Hence, if $L \to M \to N$ is an exact sequence of the category of R-modules and R-homomorphisms such that both end terms belong to δ, then M is also belong to δ. Note that the following subcategories are examples of serre subcategory of the category of the R-modules: finite R-modules; coatomic R-modules [14]; minimax R-modules [15] and trivially the zero R-modules. Generalized local cohomology was intrduced by Grothendieck [10]. They are defined as the right derived functors of the left exact functor $\Gamma_a\left(Hom_R(M,-)\right)$. Here M is a finite R-modules. They can also be computed as $H_a^i(M,N) \cong lim \xrightarrow[n\in N]{} Ext_R^i(M/a^nM,N)$ where N is an arbitrary R-modules. If $0 \to L \to P \to Q \to 0$ is an exact sequence of R-modules, then there are long exact sequence

$$0 \to H_a^0(M,L) \to H_a^0(M,P) \to H_a^0(M,Q) \to \cdots$$

$$H_a^n(M,L) \to H_a^n(M,P) \to H_a^n(M,Q) \to \cdots$$

It is well-known that the Generalized local cohomology and ordinary local cohomology modules are not in serre class of the category of R-modules in general. This notion has been studied by several authors; see, for example [1, 2, 4, 5] and [12]. Our objective in this paper is to define δ-$dim(M)$ and δ- $dim(M,N)$ such that we characterize the membership of $H_a^i(M)$ and $H_a^i(M,N)$ to δ for all $i \geq 0$. Recall that if M be an R-module and a an ideal of R, the height of a on M and the krull dimension of M with respect to δ are defined as $\delta ht_M(a) :=$ $inf\{ht_M(q)|q \in \delta - sup(M) \cap V(a)\}$ and $\delta - dim(M) := sup\{ht_M(q)|q \in \delta - sup(M)\}$ respectively.

2. SOME RESULTS IN LOCAL COHOMOLOGY AND SERRE SUBCATEGORY

The following Lemma and remark characterize membership of M to δ.

Lemma 2.1. Let $\delta \neq 0$ be a serre subcategory of the category of R-modules. If $\delta - \dim(M) = 0$ and M be a finite R-modules then $M \in \delta$.

Proof. Let M is not in δ, so by [4, Lemma 2.1], R/p is not in δ, for some $p \in Supp_R(M)$ and we have $p \in \delta - Supp_R(M)$ such that $ht_M(p) \neq 0$, consequently $\delta - \dim(M) \neq 0$ which is a contradiction.

Remark 2.2. Let $M \neq 0$ be a finite R-module. If $H_a^i(M) \in \delta$ then by [4, Lemma 2.1], $R/p \in \delta$ such that $p \in H_a^i(M) \subset Supp_R(M) \cap V(a)$, so $p \in Supp_R(M)$ and $M \in \delta$.

Definition 2.3. (See [6, Definition 2.2.1]) Let M be an R-module. The generalized ideal transform functor with respect to an ideal a of R is defined by $D_a(M,0) = \lim_{\underset{n \in N}{\longrightarrow}} Hom_R(a^n M, 0)$. Let $R^i D_a(M,0)$ denote the i-th rught derived functor of $D_a(M,0)$. One can check easily that there is a natural isomorphism $R^i D_a(M,0) \cong \lim_{\underset{n \in N}{\longrightarrow}} Ext_R^i(a^n M, 0)$.

Proposition 2.4. Let M be an R-module such that $Ext_R^i(R/a, M) \in \delta$. If $Ext_R^i\big(R/a, \Gamma_a(M)\big) \in \delta$, then $Hom_R\left(R/a, H_a^i(M)\right) \in \delta$.

Proof. The exact sequence $0 \longrightarrow \Gamma_a(M) \longrightarrow M \longrightarrow M/\Gamma_a(M) \longrightarrow 0$ induces the long exact sequence

$$\cdots \operatorname{Ext}^i_R(R/a, M) \longrightarrow \operatorname{Ext}^1_R\big(R/a, M/\Gamma_a(M)\big) \longrightarrow \operatorname{Ext}^2_R\big(R/a, \Gamma_a(M)\big) \longrightarrow \cdots$$

So $\operatorname{Ext}^1_R\big(R/a, M/\Gamma_a(M)\big) \in \delta$. We know that $\operatorname{Hom}_R\big(R/a, D_a(M)\big) \cong \big(0{:}\, D_a(M)(M)\big) \subseteq \Gamma_a\big(D_a(M)\big) = 0$ so $\operatorname{Hom}_R\big(R/a, D_a(M)\big) \in \delta$.

The exact sequence $\quad 0 \longrightarrow M/\Gamma_a(M) \longrightarrow D_a(M) \longrightarrow H^1_a(M) \longrightarrow 0$

induce the long exact sequence

$$\cdots \longrightarrow \operatorname{Hom}_R\big(R/a, D_a(M)\big) \longrightarrow \operatorname{Hom}_R\big(R/a, H^1_a(M)\big) \longrightarrow \operatorname{Ext}^1_R(R/a, M/\Gamma_a(M)) \longrightarrow \cdots$$

So $\operatorname{Hom}_R\big(R/a, H^1_a(M)\big) \in \delta$.

In [3], M. Aghapournahr and L. Melkersson gave the following condition on serre subcategory or R-modules.

Definition 2.5. Let δ be a serre subcategory of the category of R-modules. We say that δ satisfies the condition: $\quad (C_\alpha)$ if $M = \Gamma_\alpha(M)$ and if $0 : M_\alpha$ is in δ then M is in δ.

Example 2.6. The class of zero modules and artinian R-modules satisfy the condition C_α.
The following easy Lemma is useful in proof of the next theorem and we lift it to the reader.

Lemma 2.7. Let a be an ideal of R, let δ be a serre subcategory; and let $M \in \delta$. Then $\operatorname{Ext}^i_R(R/a, M) \in \delta$ for each $i \geq 0$.
The following Theorem is the main result of this section. In fact we generalize the Vanishing Theory of Grothendieck.

Theorem 2.8. Let δ be a serre subcategory of the category of R-modules. If δ satisfies the condition C_α, then $H^i_a(M) \in \delta$ for all $i > \delta - \dim M$.

Proof. When $\delta - \dim(M) = -1$, there is nothing to prove, as when $M = 0$. The result is also clear if $a = R$, as then Γ_α is the zero functor. We therefor suppose henceforth in this proof that $M \neq 0$ and a $a \subseteq m$. We argue by induction on $\delta - \dim(M)$. Let $\delta - \dim(M) = 0$. Since for each $i \geq 0$, the local cohomology functor H^i_a commutes with direct limit, and M can be viewed as the direct of its finitely generated submodules, it is sufficient for us to prove that $H^i_a(M) \in \delta$ whenever M is a finitely generated R-module, so by Lemma 2.1 and Lemma 2.7 $H^i_a(M) \in \delta$ for all $i > \delta - \dim M$. Now suppose, inductively, that $\delta - \dim M = n$ and the result has been proved for all R-modules of $\delta - \dim M$ smaller than $n > 0$. It follows that $H_a^i(M) \cong H_a^i(M/\Gamma_\alpha(M))$ for all $i > 0$. Also, $M/\Gamma_\alpha(M)$ is an a-torsion-free R-module. Then, in view of, the ideal a contains an element x which is M-regular. The exact sequence

$$0 \to M \to M \to M/xM \to 0$$

induce a long exact sequence

$$\ldots \to H_a^{i-1}(M/xM) \to H_a^i(M) \to H_a^i(M) \to \cdots$$

Since $\delta - \dim(M/xM) \leq (\delta - \dim(M)) - (\delta - ht_R(M)) = n - 1$, by induction hypothesis $H_a^{i-1}(M/xM) \in \delta$ and by the above exact sequence, we have $(0 : H_a^i(M)x) \in \delta$ so $H_a^i(M) \in \delta$ for all $i > \delta - \dim(M)$.

Remark 2.9. In Theorem 2.8 if δ satisfies the condition C_a, then $\delta - \dim(M) = \sup\{i|H_a^i(M) \in \delta\}$ and if $L \to M \to N$ be an exact sequence of finite R-modules, then $\delta - \dim(M) = \max\{\delta - \dim(N), \delta - \dim(L)\}$.

Definition 2.10. Let δ be a serre subcategory of the R-modules and let M be a module over the noetherain ring R. Following [3] an element x of R is called $\delta -$ regular on M if module $o{:}\,M\ x$ is in δ. A sequence $x_1, \ldots, x_n$ is an δ–sequence on M if is δ–regular on $M / (x_1, \ldots, x_j - 1)M$ for $j = 0,1, \ldots, n$. Also if M be a finite module such that M / aM is not in δ, where δ satisfies the condition C_a, then we denote the common length of all maximal $\delta -$ regular on M in a by $\delta - \text{depth}_a(M)$.

Example 2.11. Let δ be the class of zero modules, so $\delta - \text{depth}_a(M)$. Will be the same an ordinary $\text{depth}_a(M)$.

Corollary 2.12. Let M be a finite module and let a be an ideal of R such that M / aM is not in δ, if δ be a serre subcategory of the category of R-modules, the satisfies the condition C_a, then any integer I for which $H_a^i(M)$ is not in δ much satisfies $\delta - \text{depth}_a(M) \leq i \leq \delta - \dim(M)$.

Proof. Use Theorem 2.8 and [3, Theorem 2. 18].

Definition 2.13. (See[3, Definition 3.5]) Let δ be a serre subcategory of the category of R-module. For each R-module M, set $ch_g(a, M) = \sup\{i|H_a^i(M) \epsilon \delta\}$ which is called δ-cohomological dimension of M with respect to a. The δ-cohomological dimension of a is $ch_g(a) = \sup\{ch_g(a, M)\}$ for all R-module M.

Example 2.14. Let $\delta = \{0\}$, then $ch_g(a, M) = cd(a, M)$, and if δ, be the class of artinian modules, then $ch_g(a, M) = q(a, M)$ as in [11] and [9].

Proposition 2.15. Let a be an ideal of R. Then $ch_g(a, R) = ch_g(a)$.

Proof. Set $d = ch_g(a)$. Since R is noetherian so a can be generated by t elements. Then by [6, Theorem 3.3.1] $H_a^i(M) \epsilon \delta$ for all $i > t$, and d will be finite. Thus, $H_a^n(M) \epsilon \delta$ for all $n \geq d + 1$, and there is an R-module M with $H_a^n(M)$ is not in δ. Pick a surjective homomorphism $F \rightarrow M$, whith F a free R-module, and complete to an exact sequence $0 \rightarrow K \rightarrow F \rightarrow M \rightarrow 0$. From the resulting long exact sequence, one obtain an exact sequence $H_a^d(K) \rightarrow H_a^d(F) \rightarrow H_a^d(M) \rightarrow H_a^{d+1}(K)$. We conclude that $H_a^d(F)$ is not in δ, and so $H_a^d(R) \epsilon \delta$.

The following Theorem shows the relationship between $ch_g(M)$ and $\delta - \dim(M)$.

Theorem 2.16. Let δ be a serre subcategory of the category of R-modules. If δ satisfies the condition C_a, then $ch_g(a, M) \leq \delta - \dim(M)$.

Proof. When $ch_g(a, M) = -\infty$ or $\delta - \dim(M) = \infty$, there is nothing to prove. If $\delta - \dim(M) = 0$, then by Theorem we have $ch_g(a, M) = 0$. Now suppose, inductively, that $\delta - \dim(M) = d$ such that $d > 0$ and the result has been proved for all R-modules of δ −dimensions smaller than n. In fact we prove $H_a^i(M) \in \delta$ for all $i > n$. Let $L = M/\Gamma_a(M)$, then $l = \delta - \dim(L) \leq \delta - \dim(M) = n$, and it is sufficient for us to prove $H_a^i(L) \epsilon \delta$ for all $i > l$.

Since $\Gamma_a(L) = 0$, we have $\Gamma_a(M) = 0$ and the ideal a contains an element x which is M-regular. The exact sequence $0 \rightarrow M \rightarrow M \rightarrow M/xM \rightarrow 0$ induce a long exact sequence $H_a^{i-1}(M/xM) \rightarrow H_a^i(M) \rightarrow H_a^i(M)$. Since $\delta - \dim(M/xM) < n - 1$, it follows from the hypothesis, $H_a^{i-1}(M/xM) \in \delta$. Also, by above exact sequence and the condition C_a, $(0 :_{H_a^i(M)} x) \in \delta$, so $H_a^i(M) \in \delta$ for all $i > n$.

Corollary 2.17. Let δ be a Serre subcategory of the category of R-modules. If δ satisfies the condition C_a, then $cd_g(m) \leq \delta - \dim(R)$.

Proposition 2.18. Let δ be a Serre subcategory of the category of R-modules. If δ satisfies the condition C_a. If N be submodule of M, then $\delta - \dim(N) \leq \delta - \dim(M)$, and $\delta - \dim(M/N) \leq \delta - \dim(M)$.

Proof. Let $\delta - \dim(M) < d$. By exact sequence $0 \rightarrow N \rightarrow M \rightarrow M/N \rightarrow 0$,

We would have the long exact sequence

$$\ldots \rightarrow H_a^d(N) \rightarrow H_a^d(M) \rightarrow H_a^d\left(\frac{M}{N}\right) \rightarrow H_a^{d+1}(N) \rightarrow \cdots$$

So from Remark 2.9, $H_a^d(N) \in \delta$, and $H_a^d(M/N) \in \delta$.

3. A STUDY OF GENERALLZED LOCAL COHOMOLOGY UNDER CONDITION C_a

Lemma 3.1. *(See [8, Lemma 2.2]) Let M be an R-module. For any R-module N, there is an exact sequence,*

$$0 \to H_a^0(M,N) \to Hom_R(M,N) \to D_a(M,N) \to H_a^1(M,N) \to \ldots$$

$$\to H_a^i(M,N) \to Ext_R^i(M,N) \to R^i\, D_a(M,N) \to H_a^{1+1}(M,N) \to \cdots,$$

moreover, if M has finite projective dimension, then there is a natural isomorphism $H_a^{1+1}(M,N) \cong R^i\, D_a(M,N)$ *for all* $i \geq pdM + 1$.

Proposition 3.2. Let a, b be ideals of ring R_i such that $a \subseteq b$. Let $Ext_R^1(R/b, Hom_R(M,N)) \in \delta$, such that $Ext_R^2(R/a, H_a^0(M,N)) \in \delta$. If $Ext_R^1(M,N)) = 0$, then $Hom_R(R/b, H_a^1(M,N)))\ \delta$.

Proof. The exact sequence

$$0 \to H_a^0(M,N) \to Hom_R(M,N) \to Hom_R(M,N)/H_a^0(M,N) \to 0$$

induce the long exact sequence

$$\ldots \to Ext_R^1(R/b, Hom_R(M,N)) \to Ext_R^1(R/b, Hom_R(M,N)/H_a^0(M,N)) \to Ext_R^2(R/b, H_a^0(M,N)) \ldots,$$

then $Ext_R^1(R/b, Hom_R(M,N)/H_a^0(M,N)) \in \delta$. By Lemma 3.1, and $Ext_R^1(M,N)) = 0$, we have the exact sequence

$$0 \to Hom_R(M,N)/H_a^0(M,N) \to D_a(M,N) \to H_a^1(M,N) \to 0$$

and

$$\ldots \to Hom_R(R/b, D_a(M,N)) \to Hom_R(R/b, H_a^1(M,N)) \to$$
$$Ext_R^1\,(R/b, Hom_R(M,N)/H_a^0(M,N)) \to \cdots.$$

Since $Hom_R(R/b, D_a(M,N) \cong (0:_{D_a(M,N)} b) \subseteq \Gamma_a(D_a(M,N)) = 0, so\ Hom_R(R/b.D_a(M,N))$
$\in \delta, and\ Hom_R(R/b, H_a^i(M.N) \in \delta.$

Proposition 3.3. *Let* $Ext_R^i(R/a,n) \in \delta\ for\ all\ 0 \leq i \leq t\ if\ H_a^i(M,N) \in \delta\ for\ all\ 0 \leq i \leq t\ then\ Hom_R(R/a, H_a^i(M,N)) \in \delta.$

Proof. We use induction on t. By Lemma 2.7 for $t = 0$, proof is clear. The exact sequence

$$0 \to Hom_a^0\,(M,N)Hom_R(M,N) \to Hom_R(M,N)/H_a^0(M,N) \to 0,$$

induce a long exact sequence

$$0 \to Hom_R(R/a, H_a^0(M,N)) \to Hom_R(R/a, Hom_a(M,N)) \to \ldots$$

Suppose that $t > 0$ and that the result has bean proved for smaller values of t. It follows [6, corollary 2.1.7] that $H_a^i(N) \cong H_a^i(N/\Gamma_a(N))$ for all $i > 0$ so $\Gamma_a(N) = 0$. Let $E_R(N)$ be an injective involope of N, and $L =$

$E_R(N)/N$ so $0 \to N \to E_R(N) \to L \to O$ will be exact, and we have: $\Gamma_a\left(E_R(M)\right) = 0\,, Hom_R(R/a, E_R(M)) = 0\,, H_a^0(M, E_R(N)) = 0$. Therefor we have for all $i \geq 0$: $Ext_R^{i+1}\left(\frac{R}{a}, L\right) \cong Ext_R^{i+1}\left(\frac{R}{a}, N\right), H_a^i(M, L) \cong H_a^{i+1}(M, N),\ H_a^i\,(L) \cong H_a^{i+1}(N)$. The inductive step will be completed from the inductive hypothesis about L and this fact that $Hom_R(R/a, H_a^i(M.N)) \cong Hom_R(R/a, H_a^{i-1}(M, L))$.

Definition 3.4. Let δ be a Serre subcategory of the category of R-modules. For two R-modules M and $N\ set\ \delta - dim(M, N) = sup\{i|\ H_a^i(M, N) \in \delta\}$ which is called δ-generalized cohomological dimension of M and N with respect to a.

Theorem 3.5. If $x \in a$ be a regular element on N. Then $\delta - dim(M, N/xN) \leq \delta - dim(M.N)$.

Proof. Let $\delta - dim(M, N) = n. Then\ H_a^i(M, N) \in \delta\ for\ all\ i \geq n$. The exact sequence $0 \to N \to N \to N/xN \to 0$ implies the following long exact sequence of generalized local cohomology modules.

$$H_a^{n+1}(M, N) \to H_a^{n+1} + {}^1(M, N/xN) \to H_a^{n+2} + 2(M, N).$$

We get $H_a^{n+1}(M, N/xN) \in \delta\ and\ so\ \delta - dim(M, N/xN) \leq n$.

The following Proposition is a generalization of [13, Proposition 2.4].

Proposition3.6. Let (R, m) be a local ring such that $H_a^i(M, N) \in \delta\ for\ all\ i < n,$ $and\ \delta\ satisfies\ the\ condition\ C_a, then\ \Gamma_m\ (H_a^n(M, N)) \in \delta$.

Proof. It follows from 3.3 that $Hom_R\left(R/a, H_a^n(M, N)\right) \in \delta$. Since $\Gamma_m\left(Hom_R\left(R/a, H_a^n(M, N)\right)\right)$ is subset of $Hom_R\left(R/a, H_a^n(M, N)\right)$ then $\Gamma_m\left(Hom_R\left(R/a, H_a^n(M, N)\right)\right) \in \delta$.

By the isomorphism $\Gamma_m\left(0:_{H_a^n(M,N)} a\right) \cong \left(0:_{\Gamma_m(H_a^n(M,n))} a\right)$ and cpndition C_a , $\Gamma_m\left(H_a^n(M, N)\right) \in \delta$.

4. Open Problem

Recall that A fuzzy subset of a set *X*, we mean a function from *X* into [0,1].

A *t*-norm *T* is a function *T*: [0,1] × [0,1] → [0,1] having the following four properties:

(1) $T(x, 1) = x$,

(2) $T(x, y) \leq T(x, z)\ if\ y \leq z$,

(3) $T(x, y) = T(y, x)$,

(4) $T(x, T(y, z)) = T(T(x, y), z)$, for all $x, y, z \in [0,1]$.

An s-norm S is a function $S : [0,1] \times [0,1] \to [0,1]$ having the following four properties:

(1) $S(x,0) = x$,

(2) $S(x,y) \leq S(x,z)$ if $y \leq z$,

(3) $S(x,y) = S(y,x)$,

(4) $S(x,S(y,z)) = S(S(x,y),z)$, for all $x,y,z \in [0,1]$.

Now one can define and consider fuzzy $\delta - \dim(M)$, fuzzy $\delta - \dim(M,N)$, fuzzy $H_a^i(M)$ and fuzzy $H_a^i(M,N)$ for all $i \geq 0$, with respect to norms as First author investigated [12-23]. This can be an open problem for readers in the future.

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