A Note on a Result of Harell II - Kurata - Kröger

Pratiksha M. Kadam
Assistant Professor, Statistics Department, K. C. College,
HSNC University, Mumbai
Maharashtra, India

Abstract: *Let Ω be any non-empty open subset of $\mathbb{R}^d$, In this paper we generalise the following result of [1] for any divergence free smooth vector field V on $\mathbb{R}^d$.*
Let $f \in C^1(\Omega)$ and v be any unit vector of $R^d, B \subseteq \Omega$ open subset of R^d such that $\overline{B} \subseteq \Omega$. Let
$B_\varepsilon = \{x + \varepsilon v | x \in B\}$ and n is the outward unit normal field of Ω on $\partial\Omega$. Then

$$\lim_{t\to 0} \frac{1}{\varepsilon}\left(\int_{B_\varepsilon} f^2\, dx - \int_B f^2\, dx\right) = \int_{\partial\Omega} f^2 < v, n > ds$$

Keywords: *Weak partial derivative, Divergence,Sobolev space.*

1. Introduction

Let Ω be any open non empty subset of R^d, $u \in H^2(D)$ and V be a divergence free vector field associated with a 1-parameter group of diffeomorphisms of R^d. Now we would like to prove the result of [1] mentioned in the abstract for the vector field V.

2. Statement of the Main Result

Theorem 2.1 *Let Ω be a domain in R^d. Let $\{\psi_t\}_{t\in R}$ be 1-parameter group of diffeomorphisms of R^d associated with a divergence free vector field V.*
Let $\Omega_t = \psi_t(\Omega), (t \in R)$. Then for any $u \in H^2(D)$,

$$\lim_{t\to 0}\frac{1}{t}\left(\int_{\Omega_t} u^2\, dx - \int_\Omega u^2\, dx\right) = \int_{\partial\Omega} u^2 \langle V, \mathsf{n}\rangle dx,$$

where n denotes the outward unit normal field of the domain Ω on its boundary $\partial\Omega$.

3. Preliminaries

Let Ω be any open non empty subset of R^d. $p \geq 1$ be any real number and $L^p(\Omega)$ class of all real valued measurable functions defined on Ω for which $\int_\Omega |u(x)|^p\, dx < 0$. Let $D(\Omega)$ denote the collection of all real valued smooth functions defined on Ω having their compact support in Ω.

Definition 3.1

For $\beta = (\beta_1, \beta_2, \cdots, \beta_d) \in (Z^+)^d$, we put $|\beta| = \beta_1 + \beta_2 + \cdots + \beta_d$ and set $D^\beta = D_1^{\beta_1} D_2^{\beta_2} \cdots D_d^{\beta_d}$. A function $v \in Ł^2(\Omega)$ is said to be an 'β -th weak partial derivative' of $u \in Ł^2(\Omega)$ if

$$\int_\Omega u D^\beta\, dx = (-1)^{|\beta|} \int_\Omega v \backslash varphi\, dx \;\; \forall \varphi \in D(\Omega).$$

The β -th weak partial derivative of u, when exists, is denoted by $\partial^{\beta} u$.

Definition 3.2

For each $k \in N$, *we define Sobolev space* $H^{k,p}(\Omega)$ *by*

$$H^{k,p}(\Omega) = \{u \in L^p(\Omega) | \partial^{\beta} u \in L^p(\Omega) for |\beta| \leq k\}$$

For $u \in H^{k,p}(\Omega)$, the Sobolev norm of u is given by

$$||u||_{H^{k,p}} = \left(\sum_{|\beta| \leq k} \left|\left|\partial^{\beta} u\right|\right|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}$$

When $p = 2$, we denote $H^{k,p}(\Omega)$ by $H^k(\Omega)$. The closure of $D(\Omega)$ in $H^k(\Omega)$ is denoted by $H_0^k(\Omega)$.

Definition 3.3

Let $\boldsymbol{F}(x) = (f_1(x), f_2(x), \cdots, f_d(x))$ *be a smooth vector field defined on a bounded domain* Ω *in* R^d. *Then the divergence* $div\boldsymbol{F}$ *of the vector field* $\boldsymbol{F}$ *is the function defined on* Ω *by*

$$(div\boldsymbol{F})(x) = \sum_{j=1}^{d} \frac{\partial f_j}{\partial x_j}(x), \quad (x \in \Omega).$$

$\boldsymbol{F}$ *is said be a divergence free vector field on a domain* Ω, *if* $div\boldsymbol{F}(x) = 0 \ \forall \ x \in \Omega$

4. Some Required Results

Lemma 4.1

Let Ω *be a non-empty open subset of* R^d *and let* $u \in H^2(\Omega)$. *Then* $u^2 \in H^{1,1}(\Omega)$

Proof:

$$\int_{\Omega} u^2 \, dx = ||u^2||_{L^1(\Omega)} < \infty u^2 \in L^1(\Omega) \tag{1}$$

$$\therefore \int_{\Omega} \left| \frac{\partial u^2}{\partial x_i} \right| dx = 2 \int_{\Omega} \left| u \frac{\partial u}{\partial x_i} \right| dx$$

$$\leq 2||u||_{L^2(\Omega)} || \frac{\partial u}{\partial x_i} ||_{L^2(\Omega)}$$

$$\leq 2||u||^2_{H^2(\Omega)} < \infty.$$

$$\therefore \frac{\partial u^2}{\partial x_i} \in L^1(\Omega) \forall i = 1,2, \cdots, d. \tag{2}$$

From (1) and (2), $u^2 \in H^{1,1}(\Omega)$.

Lemma 4.2

Let $\{\psi_t\}_{t \in R}$ *be the 1-parameter group of diffeomorphisms of* R^d *associated with the vector field* V. *Let* $j_t(x) = det(D\psi_t(x))$. *Then* $j_t^{'}(x)|_{t=0} = divV(x)$.

Proof:

$$j_t(x) = det(D\psi_t(x))$$

$$V(x) = (v_1(x), v_2(x), \cdots, v_d(x))$$

$$\psi_t(x) = x + tV(x) + O_2(t)$$

$$\therefore D\psi_t(x) = Id_{d\times d} + tDV(x) + Higher\ order\ terms\ in\ t$$

$$\therefore j_t(x) = det(Id_{d\times d} + tDV(x) + \cdots)$$

$$\therefore j_t(x) = det\begin{pmatrix} 1+\frac{\partial v_1}{\partial x_1}+\cdots, & \frac{\partial v_1}{\partial x_2}+\cdots, & \cdots, & \frac{\partial v_1}{\partial x_d}+\cdots \\ \frac{\partial v_2}{\partial x_1}+\cdots, & 1+\frac{\partial v_2}{\partial x_2}+\cdots, & \cdots, & \frac{\partial v_2}{\partial x_d}+\cdots \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial v_d}{\partial x_1}+\cdots, & \frac{\partial v_d}{\partial x_2}+\cdots, & \cdots, & 1+\frac{\partial v_d}{\partial x_d}+ \end{pmatrix} = \sum_{j=1}^{d}\frac{\partial v_j}{\partial x_j}$$

$$\therefore j_t(x) = divV(x).$$

5. Proof Of Theorem 2.1

Proof:

$$div(u^2V) = \sum_{j=1}^{d}\frac{\partial(u^2v_j)}{\partial x_j}$$

$$= \sum_{j=1}^{d}\left(u^2\frac{\partial v_j}{\partial x_j} + v_j\frac{\partial u^2}{\partial x_j}\right)$$

$$= u^2\sum_{j=1}^{d}\frac{\partial v_j}{\partial x_j} + \sum_{j=1}^{d}v_j\frac{\partial u^2}{\partial x_j}$$

$$= u^2 divV + \langle \nabla u^2, V\rangle$$

$$= \langle \nabla u^2, V\rangle \qquad (\because divV = 0)$$

$$\therefore div(u^2V) = \langle \nabla u^2, V\rangle \tag{3}$$

Let $j_t(x) := det(D\psi_t)(x), (x \in R^d)$. Then $j_0(x) = 1 \quad \forall\ x \in R^d$ and by lemma 4.2 $j_{t'}(x) := \frac{dj_t}{dt}|_{t=0}(x) = div(V(x))$. (4)

By lemma 4.1 $u^2 \in H^{1,1}(\Omega)$. By the change of variable formula for integration

$$\lim_{t\to 0}\frac{1}{t}\left(\int_{\Omega_t}u^2dx - \int_{\Omega}u^2dx\right) = \lim_{t\to 0}\int_{\Omega}\frac{(u^2\circ\psi_t)j_t - u^2}{t}dx. \tag{5}$$

By Dominated Convergence theorem,

$$\lim_{t\to 0}\frac{1}{t}\left(\int_{\Omega_t} u^2\,dx-\int_{\Omega} u^2\,dx\right)$$
$$=\int_{\Omega}\lim_{t\to 0}\left(\frac{(u^2\circ\psi_t)j_t-u^2}{t}\right)dx$$
$$=\int_{\Omega}(u^2\circ\psi_t)'j_0+j_t'u^2\circ\psi_0)dx \tag{6}$$

Claim 5.1: $(u^2\circ\psi_t)'|_{t=0}=<\nabla u^2,V>$ *almost everywhere on* Ω.

Proof of claim 5.1:
By lemma 4.1 $u^2\in H^{1,1}(\Omega)$
$$\frac{d}{dt}(u^2\circ\psi_t)|_{t=0}=Du_x^2\left(\frac{d}{dt}\psi_t\right)|_{t=0} \tag{7}$$
$$=Du_x^2(V(x)) \tag{8}$$
$$=<\nabla u^2,V>(x) \tag{9}$$
Hence the claim.

Claim 5.2 $\frac{d}{dt}\big((u^2\circ\psi_t)j_t\big)|_{t=0}=<\nabla u^2,V>(x)+u^2(x)div\big(V(x)\big)$.

Proof of claim 5.2:
$$\frac{d}{dt}\big((u^2\circ\psi_t)j_t\big)|_{t=0}$$
$$=j_0(x)\frac{d}{dt}(u^2\circ\psi_t)(x)|_{t=0}+(u^2\circ\psi_0)(x)\frac{d}{dt}j_t(x)|_{t=0}$$
$$=\frac{d}{dt}(u^2\circ\psi_t)(x)|_{t=0}+u^2j_t'(x)|_{t=0}\qquad(\because j_0(x)=1)$$

Hence by claim 5.1 and lemma 4.2,
$$\frac{d}{dt}\big((u^2\circ\psi_t)j_t\big)|_{t=0}=<\nabla u^2,V>(x)+u^2(x)div\big(V(x)\big).$$
Thus from claim 5.1 ,claim 5.2 and (3)
$$\lim_{t\to 0}\frac{1}{t}\left(\int_{\Omega_t}u^2\,dx-\int_{\Omega}u^2\,dx\right)=\int_{\Omega}<\nabla u^2,V>dx$$
Thus by Gauss Divergence Theorem,
$$\lim_{t\to 0}\frac{1}{t}\left(\int_{\Omega_t}u^2\,dx-\int_{\Omega}u^2\,dx\right)=\int_{\partial\Omega}u^2<V,n>ds.$$
Hence the theorem.

6. Conclusion

The theorem in [1] is true for the general vector field with the suitable assumptions (i.e. Divergence free vector field).

7. References

[1] Harrell II, Kröger P., Kurata K., *On the placement of an obstacle or a well so as to optimize the fundamental eigenvalue.*, SIAM journal on Mathematical Analysis, Vol. 33, 1, (2001)