On The Quintic Equation With Five Unknowns $x^4 - y^4 = 34(z^2 - w^2)T^3$

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Abstract: ***We obtain infinitely many non-zero integer satisfying the quintic equation. Various interesting properties among the values of and T are presented.***

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Introduction

The theory of Diophantine equations offers a rich variety of fascinating problems. In particular, quintic equations homogeneous or non-homogeneous have aroused the interest of numerous mathematicians since antiquity [1-4]. For illustration, one may refer [5-10] for quintic equation with five unknowns. This paper concerns with the problems of determining non-trivial integral solutions of the non-homogeneous quintic equation with five unknowns given by $x^4 - y^4 = 34(z^2 - w^2)T^3$. A few relations between the solutions and special numbers are presented.

Notation:

$t_{m,n}$ **:** Polygonal number of rank n with size m

p_n^m : Pyramidal number of rank n with size m

CP_n^m : Centered Pyramidal number of rank n with size m

$CP_{m,n}$: Centered number of rank n with size m

SO_n : Stella Octangular number of rank n

S_n : Star number of rank n

PR_n : Pronic number of rank n

OH_n : Octahedral number of rank n

Method of analysis:

The quintic equation with five unknowns to be solved is

$$x^4 - y^4 = 34(z^2 - w^2)T^3 \quad (1)$$

The process of obtaining patterns of integral solutions to (1) are illustrated below.

Pattern 1

Introduction of the transformations

$$x = u + v,\ \ y = u - v,\ \ z = 2u + v,\ w = 2u - v \qquad (2)$$

In (1) leads to

$$u^2 + v^2 = 34T^3 \qquad (3)$$

Let $T = a^2 + b^2$ (4)

Write 34 as $34 = (5+3i)(5-3i)$ (5)

Using (4) and (5) in (3) and applying the method of factorization, define

$$u + iv = (5+3i)(a+ib)^3 \qquad (6)$$

Equating real and imaginary parts we have,

$$\left.\begin{aligned} u &= 5a^3 - 9a^2b - 15ab^2 + 3b^3 \\ v &= 3a^3 + 15a^2b - 9ab^2 - 5b^3 \end{aligned}\right\} \qquad (7)$$

Using (7) in (2), the corresponding non-zero distinct integral solutions to (1) are given by

$$x = 8a^3 + 6a^2b - 24ab^2 - 2b^3$$

$$y = 2a^3 - 24a^2b - 6ab^2 + 8b^3$$

$$z = 13a^3 - 3a^2b - 39ab^2 + b^3$$

$$w = 7a^3 - 33a^2b - 21ab^2 + 11b^3$$

$$T = a^2 + b^2$$

Properties:

- $4x(a,1) + y(a,1) - 68CP_{3,a} \equiv 0 \pmod 8$
- $11y(a,1) - 8w(a,1) + 68CP_{3,a} \equiv 0 \pmod 8$
- $11z(a,1) - w(a,1) - 272P_a^5 + 272t_{3,a} \equiv 0 \pmod 8$
- $11y(a,1) - 8w(a,1) + 204P_{a-1}^3 \equiv 0 \pmod 4$
- $11y(a,1) - 8w(a,1) + 68P_a^5 - 68t_{3,a} \equiv 0 \pmod 4$
- $4x(a,1) + y(a,1) - 102P_a^4 + 102t_{3,a} \equiv 0 \pmod 5$
- $24z(a,1) - 3y(a,1) - 306CP_{6,a} \equiv 0 \pmod 9$
- $4x(a,1) + y(a,1) + 34CP_{6,a} \equiv 0 \pmod 6$
- $24z(a,1) - 3y(a,1) - 2CP_{6,a} - 456CP_{4,a} \equiv 0 \bmod (107)$
- $4x(a,1) + y(a,1) - 68P_a^5 + t_{42,a} + t_{30,a} \equiv 0 \pmod{67}$
- $4x(a,1) + y(a,1) - 204P_{a-1}^3 \equiv 0 \pmod{34}$
- $11z(a,1) - w(a,1) - 272P_a^5 + t_{202,a} + t_{74,a} \equiv 0 \pmod 2$
- $11z(a,1) - w(a,1) - 816P_{a-1}^3 \equiv 0 \pmod 4$

- $11z(a,1)-w(a,1)-272P_a^5+t_{274,a}\equiv 0(\bmod 181)$
- $4x(a,1)+y(a,1)-51CP_{4,a}\equiv 0(\bmod 17)$
- $4x(a,1)+y(a,1)-68P_a^5+t_{70,a}\equiv 0(\bmod 5)$
- $11y(a,1)-8w(a,1)+68P_a^5+t_{70,a}\equiv 0(\bmod 3)$
- $24z(a,1)-3y(a,1)-612P_a^5+612t_{3,a}\equiv 0(\bmod 36)$
- $11y(a,1)-8w(a,1)+34CP_{6,a}\equiv 0(\bmod 17)$
- $24z(a,1)-3y(a,1)-CP_{6,a}-366CP_{5,a}\equiv 0(\bmod 89)$

Pattern 2

Write 34 as $34=(3+5i)(3-5i)$ (8)

Following the procedure similar to pattern (1), the non-zero distinct integer solution of (1) is given by

$$x=8a^3-6a^2b-24ab^2+2b^3$$
$$y=-2a^3-24a^2b+6ab^2+8b^3$$
$$z=11a^3-21a^2b-33ab^2+7b^3$$
$$w=a^3-39a^2b-3ab^2+13b^3$$
$$T=a^2+b^2$$

Properties:

- $4x(a,1)-y(a,1)-17SO_a\equiv 0(\bmod 17)$
- $4x(a,1)-y(a,1)-68CP_{3,a}\equiv 0(\bmod 8)$
- $4x(a+1,a)-y(a+1,a)+34SO_a\equiv 0(\bmod 34)$
- $4x(a,1)-y(a,1)-34CP_{6,a}\equiv 0(\bmod 51)$
- $4x(a,1)-y(a,1)-17SO_a\equiv 0(\bmod 17)$
- $4x(a,a+1)-y(a,a+1)+136P_a^5+136PR_a\equiv 0(\bmod 17)$
- $4x(a,a)-y(a,a)+34SO_a\equiv 0(\bmod 2)$
- $11y(1,b)+2z(1,b)-153OH_a\equiv 0(\bmod 21)$
- $2z(a,a)+11y(a,a)+204CP_{6,a}=0$
- $11y(a,a+1)+2z(a,a+1)+408CP_{3,a}\equiv 0(\bmod 102)$
- $z(a,a)-11w(a,a)-204CP_{8,a}\equiv 0(\bmod 17)$
- $13z(a,1)-7w(a,1)-204P_a^6+t_{206,a}\equiv 0(\bmod 19)$
- $13z(a,a)-7w(a,a)+96CP_{17,a}\equiv 0(\bmod 11)$

- $z(1,b)-11w(1,b)\equiv 0(\mathrm{mod}\,136)$

Pattern 3

In addition to (8),

Write 34 as $34=\frac{(11+27i)(11-27i)}{5^2}$ (9)

Following the procedure similar to Pattern:1, we have,

$$\left.\begin{aligned} u&=\frac{1}{5}[11a^3-81a^2b-33ab^2+27b^3 \\ v&=\frac{1}{5}[27a^3+33a^2b-81ab^2-11b^3] \end{aligned}\right\} \quad (10)$$

Thus, replacing a by 5Aand b by 5B in (10) the corresponding non-zero distinct integral solutions to (1) are given by

$$x=950A^3-1200A^2B-2850AB^2+400B^3$$
$$y=-400A^3-2850A^2B+1200AB^2+950B^3$$
$$z=1225A^3-3225A^2B-3675AB^2+1075B^3$$
$$w=-125A^3-4875A^2B+375AB^2+1625B^3$$
$$T=25A^2+25B^2$$

Properties:

- $19x(A,A)-8y(A,A)+10200CP_{25,A}\equiv 0(\mathrm{mod}\,19)$
- $8x(A,1)+19y(A,1)-6375PR_A\equiv 0(\mathrm{mod}\,51)$
- $5y(A,A)-16w(A,A)-12750CP_{20,A}\equiv 0(\mathrm{mod}\,25)$
- $65x(A,1)-16w(A,1)+22500CP_{17,A}\equiv 0(\mathrm{mod}\,8)$
- $8x(A,1)+19y(A,1)-1062S_A-t_{8,A}\equiv 1(\mathrm{mod}\,2)$
- $16w(A+1,1)-65x(A+1,1)-127500CP_{3,A}+382500t_{3,A}\equiv 0(\mathrm{mod}\,2)$
- $8x(A,A^2)-19y(A,A^2)-6375t_{4,A^2}+1080t_{4,A^3}=0$
- $49z(a+1,a)+5w(a+1,a)+170000CP_{6,A}\equiv 0(\mathrm{mod}\,2)$

Pattern 4

It is worth to note that 34 may also be written as

$$34=\frac{(27+11i)(27-11i)}{5^2} \quad (11)$$

For this choice the corresponding non-zero distinct integral solution to (1) are found to be

$$x = 950A^3 + 1200A^2B - 2850AB^2 - 400B^3$$
$$y = 400A^3 - 2850A^2B - 1200AB^2 + 950B^3$$
$$z = 1625A^3 + 375A^2B - 4875AB^2 - 125B^3$$
$$w = 1075A^3 - 3675A^2B - 3225AB^2 + 1225B^3$$
$$T = 25A^2 + 25B^2$$

Property

- $x(A,1) - y(A,1) - 1100P_A^5 - 275S_A - 1850t_{4,A} \equiv 0 \pmod 5$
- $y(A,1) - z(A,1) + 2450P_A^5 + 500t_{10,A} - 4350t_{3,n} + 2175t_{4,A} \equiv 0 \pmod 5$
- $x(A,1) - z(A,1) + 150CP_{27,A} - 4050t_{3,A} + 1725t_{4,A} \equiv 0 \pmod 5$
- $w(A,1) + T(A,1) - Z(A,1) - 525CP_{3,A} + 150CP_{30,A} + 4350t_{4A} - 2100t_{3,A} \equiv 0 \pmod 5$

Pattern 5

Write (3) as $u^2 + v^2 = 34T^3 * 1$ (12)

Write 1 as $\frac{(3-4i)(3+4i))}{5^2}$ (13)

Substituting (4), (5) and (13) in (12) and employing the factorization method, define

$$u + iv = (5+3i)(a+ib)^3 (\frac{3+4i}{5}) \quad (14)$$

Equating real and imaginary parts, we get

$$\left.\begin{aligned} u &= \frac{1}{5}[3a^3 - 9ab^2 - 87a^2b + 29b^3] \\ v &= \frac{1}{5}[29a^3 + 9a^2b - 87ab^2 - 3b^3] \end{aligned}\right\} \quad (15)$$

Using (15) in (2), we have

$$\left.\begin{aligned} x &= 800A^3 - 2400AB^2 - 1950A^2B + 650B^3 \\ y &= -650A^3 - 2400A^2B + 1950AB^2 + 800B^3 \\ z &= 875A^3 - 2625AB^2 - 4125A^2B + 1375B^3 \\ w &= -575A^3 + 1725AB^2 - 4575A^2B + 1525B^3 \\ T &= 25A^2 + 25B^2 \end{aligned}\right\} \quad (16)$$

Thus, (16) represent the non-zero distinct integral solutions to (1)

Remark:

Equation (13) may also be expressed in seven different cases that are presented below:

Case (i)

$$1=\frac{(4-3i)(4+3i)}{5^2} \tag{17}$$

Case (ii)

$$1=\frac{(5+12i)(5-12i)}{13^2} \tag{18}$$

Case (iii)

$$1=\frac{(12+5i)(12-5i)}{13^2} \tag{19}$$

Case (iv)

$$1=\frac{(6+8i)(6-8i)}{10^2} \tag{20}$$

Case (v)

$$1=\frac{(8+6i)(8-6i)}{10^2} \tag{21}$$

Case (vi)

$$1=\frac{(9+12i)(9-12i)}{15^2} \tag{22}$$

Case (vii)

$$1=\frac{(12+9i)(12-9i)}{13^2} \tag{23}$$

Following the procedure as presented in Pattern.5 the corresponding non-zero distinct integral solutions to (1) are presented below

Solution for case (i)

$$x = 950A^3 - 1200A^2B - 2850AB^2 + 400B^3$$
$$y = -400A^3 - 2850A^2B + 1200AB^2 + 950B^3$$
$$z = 1225A^3 - 3225A^2B - 3675AB^2 + 1075B^3$$
$$w = -125A^3 - 4875A^2B + 375AB^2 + 1625B^3$$
$$T = 25A^2 + 25B^2$$

Solution for case (ii)

$$x = 10816A^3 - 43602A^2B - 32448AB^2 + 14534B^3$$
$$y = -14534A^3 - 32448A^2B + 43602AB^2 + 10816B^3$$
$$z = 8957A^3 - 81627A^2B - 26871AB^2 + 27209B^3$$
$$w = -16393A^3 - 70473A^2B + 49179AB^2 + 23491B^3$$
$$T = 169A^2 + 169B^2$$

Solution for case (iii)

$$x = 17914A^3 - 8112A^2B - 53742AB^2 + 2704B^3$$
$$y = -2704A^3 - 53742A^2B + 8112AB^2 + 17914B^3$$
$$z = 25519A^3 - 39039A^2B - 76557AB^2 + 13013B^3$$
$$w = 4901A^3 - 84669A^2B - 14703AB^2 + 28223B^3$$
$$T = 169A^2 + 169B^2$$

Solution for case (iv)

$$x = 6400A^3 - 15600A^2B - 19200AB^2 + 5200B^3$$
$$y = -5200A^3 - 19200A^2B + 15600AB^2 + 6400B^3$$
$$z = 7000A^3 - 33000A^2B - 21000AB^2 + 11000B^3$$
$$w = -4600A^3 - 36600A^2B + 13800AB^2 + 12200B^3$$
$$T = 100A^2 + 100B^2$$

Solution for case (v)

$$x = 7600A^3 - 9600A^2B - 22800AB^2 + 3200B^3$$
$$y = -3200A^3 - 22800A^2B + 9600AB^2 + 7600B^3$$
$$z = 9800A^3 - 25800A^2B - 29400AB^2 + 8600B^3$$
$$w = -1000A^3 - 39000A^2B + 3000AB^2 + 13000B^3$$
$$T = 100A^2 + 100B^2$$

Solution for case (vi)

$$x = 21600A^3 - 52650A^2B - 64800AB^2 + 17550B^3$$
$$y = -17550A^3 - 64800A^2B + 52650AB^2 + 21600B^3$$
$$z = 23625A^3 - 111375A^2B - 70875AB^2 + 37125B^3$$
$$w = -15525A^3 - 123525A^2B + 46575AB^2 + 41175B^3$$
$$T = 225A^2 + 225B^2$$

Solution for case (vii)

$$x = 25650A^3 - 32400A^2B - 76950AB^2 + 10800B^3$$
$$y = -10800A^3 - 76950A^2B + 32400AB^2 + 25650B^3$$
$$z = 33075A^3 - 87075A^2B - 99225AB^2 + 29025B^3$$
$$w = -3375A^3 - 131625A^2B + 10125AB^2 + 43875B^3$$
$$T = 225A^2 + 225B^2$$

Conclusion

In this paper, we have presented different choices of integer solutions to the quintic equation with five unknowns $x^4 - y^4 = 34(z^2 - w^2)T^3$ To conclude, as quintic equations are rich in variety, One may consider other forms of quintic equations and their solutions and corresponding properties.

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